Single Machine With Wiener Increment Yield: Optimal Offline Control

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***Abstract*—In many manufacturing systems prone to random interference, such as machine breakdowns and fluctuating yield, information about the interference as well as other system states, e.g., inventory levels, may not be available in a timely manner. Therefore an online feedback control will be difficult to implement, and a good offline control strategy may be the only alternative. In this note we develop such a strategy for a production system with random yield, which is characterized by a Wiener process. Assuming the initial inventory level is known, we derive closed form expressions of the optimal production control to minimize the expected inventory and backlog costs over a production horizon.**

***Index Terms*—Maximum principle, offline control, random yield.**

I. Introduction

The stochastic production control in a product defect or machine failure prone environment is typically considered the prerogative of real-time or online approaches (see, for example, pioneering work of Kimemia [6], Kimemia and Gershwin [7], and Akella and Kumar [1]). The optimal production rate $u(t)$, which minimizes the expected inventory holding and backlog costs, is usually a function of the inventory $X(t)$. To prove the optimality of the control, certain assumptions will have to be asserted, e.g., the observability of the inventory level and machine states, and notably the Markovian supposition that stipulates that the transition between an operational state to a breakdown state of the machine is described by a continuous-time Markov chain.

Unfortunately, in certain manufacturing systems, the information about either machine states or inventory levels may at best be imprecise, if not unobtainable. One example is the chip fabricating facility, where yield or machine breakdowns are due to complex causes which are difficult to identify. The system, like many modern ones, could continue producing at the same rate even when it has been malfunctioning, because it is the part inspection, at a much later stage of the production, that will eventually unveil the culprits.

It is also commonplace in some production systems that inventory levels are not obtainable continuously. This reality, in conjunction with the often ambiguous machine states described above, warrants the exploration of an offline control methodology, which provides a better system management when the above-mentioned information is absent.

Such an optimal offline control scheme is developed in this note for a production system with random yield and constant demand. Random yields in various forms have been considered by many authors. Comprehensive literature reviews on stochastic manufacturing flow control and lot sizing with random yields or unreliable machines can be found in Haurie [4] and Yano and Lee [12]. In addition, make-to-order batch manufacturing with random yield is considered by Gerchak and Grosfeld-Nir (1998) and Wang and Gerchak [11]. In these papers it is proven that the optimal policy is of threshold control type—stop if and only if the stock is larger than some critical value. Gerchak and Grosfeld-Nir (1998) develop a computer program for solving the problem of binomial yields, while Wang and Gerchak [11] study the critical value for different production cases. The optimal control derived in this note is significantly different from the traditional threshold control expected under the Markovian assumption, which alternates between zero and the maximum production rate. Indeed, the production rate is not necessarily maximal when the expected inventory level is less than the critical value X^*, nor is it necessarily zero when the inventory level is larger than X^*.

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II. Problem Statement

Consider a single machine, single part-type production system with random yield characterized by a Wiener process. Similar to the Wiener-increment-based stochastic production models [4], the inventory level $X(t)$ is described by the following stochastic differential equation:

$$dX(t) = \big(Pdt + \overline{\beta} d\mu(t)\big)\, u(t) - Ddt \tag{1}$$

where $X(0)$ is a given deterministic initial inventory and $u(t)$ is the production rate

$$0 \le u(t) \le U \tag{2}$$

$P, 0 < P < 1 (U > D/P)$, is the average yield—the proportion of the good parts produced, $\mu(t)$ is a Wiener process, $\overline{\beta}$ is the variability constant of the yield, $d\mu(t)$ is the Wiener increment, and D is the constant demand rate.

Similar to Shu and Perkins [10] and Khmelnitski and Caramanis [5], we consider a quadratic inventory cost which is incurred when either $X(t) > 0$ (inventory surplus), or $X(t) < 0$ (shortage). The objective of the production control is to minimize the overall expected inventory cost

$$J = E\left[\int_0^T X^2(t)dt\right] \tag{3}$$

subject to (1) and (2), where T is the planning horizon during which the state of the system can be evaluated.

III. Equivalent Deterministic Formulation

To find the optimal offline control, we introduce an equivalent deterministic formulation.

Lemma 1: Problem (1)–(3) is equivalent to minimizing

$$J = \int_0^T \left(\left[X(0) - Dt + P \int_0^t u(s)ds \right]^2 + \beta \int_0^t u^2(s)ds \right) dt \tag{4}$$

subject to (2), where $\beta = \overline{\beta}^2$.

Proof: Integrating (1), we have

$$X(t) = X(0) - Dt + \int_0^T Pu(s)ds + \int_0^T \overline{\beta} u(s) d\mu(s) \tag{5}$$

which leads to

$$X^2(t) = [X(0) - Dt]^2 + 2\,[X(0) - Dt]\, L(t) + [L(t)]^2 \tag{6}$$

where $L(t) = \int_0^t Pu(s)ds + \int_0^t \overline{\beta}u(s)d\mu(s)$. Using the fact that the expectation of the stochastic (Ito) integrals is zero, we obtain

$$E\left[X^2(t)\right] = [X(0) - Dt]^2 + 2[X(0) - Dt]\int_0^t Pu(s)ds + E\left[\int_0^t Pu(s)ds + \int_0^t \overline{\beta}u(s)d\mu(s)\right]^2. \quad (7)$$

With respect to the Ito isometry, $E[\int_0^t A(\tau)dW(\tau)]^2 = \int_0^t E[A^2(\tau)]d\tau$ [8], the last term in (7) can be rewritten as

$$\begin{aligned} E&\left[\int_0^t Pu(s)ds + \int_0^t \overline{\beta}u(s)d\mu(s)\right]^2 \\ &= E\left[\left(\int_0^t Pu(s)ds\right)^2 + 2\int_0^t Pu(s)ds\int_0^t \overline{\beta}u(s)d\mu(s) + \left(\int_0^t \overline{\beta}u(s)d\mu(s)\right)^2\right] \\ &= P^2\left[\int_0^t u(s)ds\right]^2 + \overline{\beta}^2\int_0^t u^2(s)ds. \end{aligned}$$

Therefore, we have

$$\begin{aligned} J &= E\left[\int_0^T X^2(t)dt\right] \\ &= \int_0^T E\left[X^2(t)\right]dt \\ &= \int_0^T \left([X(0) - Dt]^2 + 2[X(0) - Dt]P\int_0^t u(s)ds + P^2\left[\int_0^t u(s)ds\right]^2 + \overline{\beta}^2\int_0^t u^2(s)ds\right)dt. \end{aligned}$$

Finally, by rearranging the terms in the last expression and using $\beta = \overline{\beta}^2$, we arrive at (4). ■

We use the maximum principle to solve the problem [9]. Note that the objective function (4) is a summation of strictly convex functions. This implies that the problem has a unique optimal solution.

The objective function (4) contains integrals over independent variable t, thus it does not satisfy the canonical optimal control formulation needed for using the maximum principle. Hence, we introduce the expected inventory, $X_E(t)$, which satisfies

$$\dot{X}_E(t) = Pu(t) - D,\ X_E(0) = X(0) \quad (8)$$

and the cumulative quadratic control, $Y(t)$, which satisfies

$$\dot{Y}(t) = u^2(t),\ Y(0) = 0. \quad (9)$$

Then the objective function (4) takes the following form:

$$J = \int_0^T \left[X_E^2(t) + \beta Y(t)\right]dt \rightarrow \min. \quad (10)$$

Formulation (2), (8)–(10) is canonical. According to the maximum principle, the control $u(t)$ which maximizes the Hamiltonian $H(t)$ subject to constraint (2) is optimal for (8)–(10), and thus is optimal for the original problem. The Hamiltonian is defined as

$$H(t) = -X_E^2(t) - \beta Y(t) + \psi_X(t)(Pu(t) - D) + \psi_Y(t)u^2(t) \quad (11)$$

where the costate variables $\psi_X(t)$ and $\psi_Y(t)$ satisfy the following costate equations

$$\dot{\psi}_X(t) = 2X_E(t),\ \psi_X(T) = 0; \quad (12)$$

$$\dot{\psi}_Y(t) = \beta,\ \psi_Y(T) = 0. \quad (13)$$

IV. Two Special Cases of the Optimal Solution

As delineated here, depending upon the level of the initial inventory $X(0)$, different optimal control formulations will have to be employed. The formulations are, unfortunately, rather involved, and their proofs convoluted. To make the results more comprehensible, we will start off by proving two special cases.

A. The First Special Case: $X(0) \geq DT$

In this case, the initial inventory is large enough to meet the demand for the entire planning horizon T. Therefore, the optimal policy, as one expects, is not to produce at all.

Lemma 2: If $X(0) \geq DT$, then $u(t) = 0, 0 \leq t < T$ is optimal.

Proof: Since $X_E(0) \geq DT$ means $X_E(t) = X(0) + \int_0^t (Pu(\tau) - D)d\tau > 0$, we have $\dot{\psi}_X(t) = 2X_E(t) > 0, 0 \leq t < T$. But $\psi_X(T) = 0$, therefore $\psi_X(t) < 0$ and $\psi_Y(t) < 0, 0 \leq t < T$ (see (12) and (13), respectively). Therefore $u(t) = 0, 0 \leq t < T$ maximizes (11) and thus optimal. ■

B. The Second Special Case: $X(0)$ is Moderately Large, but $X(0) < dt$

As shown in Theorem 1, given two critical values, $X^* = -\beta D/P^2$ and $\hat{X}$ which can be evaluated through equations depending on the system and initial conditions, $\hat{X} > X^*$, we will have a three-phase control when $\hat{X} > X(0) \geq X^*$ [see Fig. 1(a)]. Initially the optimal production rate $u(t)$ is zero, and thus the average inventory level $X_E(t)$ decreases. This is the first phase, which is identical to the control in the first special case. At a time point t_ψ (a certain level of $X_E(t)$, $\hat{X} > X_E(t_\psi) > X^*$), the optimal $u(t)$ becomes positive but is still small enough so that $X_E(t)$ continues its decline. This is the second phase. Finally, as soon as $X_E(t)$ reaches a critical value, X^*, (this time point is referred to as t_O), the optimal $u(t)$ becomes a constant, $u^*(t) = D/P$ and from that point on $X_E(t)$ and $u^*(t)$ will remain to be X^* and D/P, respectively. This is the third phase during which the system enters the steady state. The optimal control when $X(0)$ is smaller than X^* is the mirror image of the described control (see Fig. 1(b)) and therefore is not considered in the note. On the other hand, if $DT > X(0) \geq \hat{X}$, then the optimal control will include only the first two phases. Note, that the proofs of the equation for $\hat{X}$ and the existence of t_O when $\hat{X} > X(0) \geq X^*$, which utilize the asymptotic behaviors of the family of Bessel functions, are tedious and therefore excluded. To prove Theorem 1, we first need to establish the following lemma.

Lemma 3: Assume functions $\psi(t)$, $X(t)$ and $u(t) = \begin{cases} 0, & \psi(t) < 0 \\ (P\psi(t))/(2\beta(T-t)), & \psi(t) \geq 0 \end{cases}$ satisfy $\dot{X}(t) = Pu(t) - D$ and $\psi(t) = 2X(t)$ for $0 \leq t \leq T$ where $\beta > 0$, $P > 0$ and $D > 0$ are constants. Furthermore, assume $\psi(T) = 0$, $X(0) > \widetilde{X} = -\beta D/P^2$ and $X(t') = \widetilde{X}$ for some t', $0 \leq t' \leq T$ and $X(t) \neq \widetilde{X}$, $0 \leq t < t'$. Then

1) $\psi(t) \leq -2\widetilde{X}(T-t)$, $0 \leq t \leq t'$;
2) $X(t) = \widetilde{X}$ and $\psi(t) = -2\widetilde{X}(T-t)$ for $t' \leq t \leq T$.

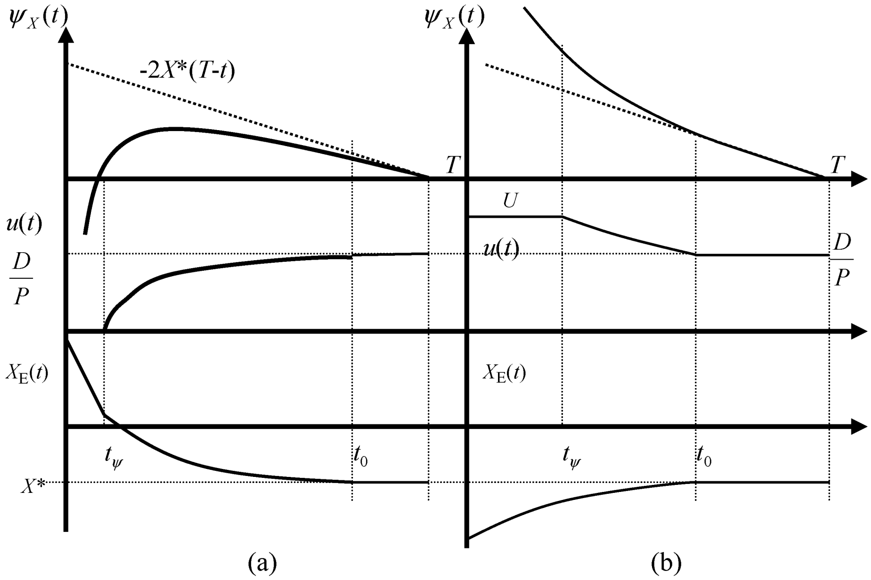

Fig. 1. Optimal behavior of the system for (a) $X(0) > X^*$ and (b) $X(0) < X^*$.

Proof: We first show that $\psi(t) < -2\widetilde{X}(T-t)$, $0 \le t < t'$. Since $X(0) > \widetilde{X}$, $X(t') = \widetilde{X}$ and $X(t) \ne \widetilde{X}$, $0 \le t < t'$, we must have $X(t) > \widetilde{X}$, $0 \le t < t'$. Thus, there is a t'', $t'' < t'$, such that $\dot{X}(t) = Pu(t) - D < 0$ for $t'' \le t < t'$. Therefore, $u(t) < (D/P)$, $t'' \le t < t'$, which leads to

$$\psi(t) < -2\widetilde{X}(T-t) \text{ for } t'' \le t < t'. \tag{14}$$

If $\psi(\bar{t}) > -2\widetilde{X}(T-\bar{t})$ for some $\bar{t}$, $0 \le \bar{t} < t''$, then because $\dot{\psi}(t) = 2X(t) > 2\widetilde{X}$ for $0 \le t < t'$, we would have $\psi(t'') > -2\widetilde{X}(T-t'')$. However, this contradicts (14). Therefore, $\psi(t) < -2\widetilde{X}(T-t)$, for $0 \le t < t'$.

We now show that $\psi(t') = -2\widetilde{X}(T-t')$. Assume the opposite were true, that is, $\psi(t') < -2\widetilde{X}(T-t')$. Thus, $u(t') < (D/P)$ and $\dot{X}(t') < 0$. Therefore, there would exist a t''', $t' < t''' < T$ such that $X(t) < \widetilde{X}$ for $t' < t \le t'''$. Thus, $\dot{\psi}(t) = 2X(t) < 2\widetilde{X}$ and $\psi(t) < -2\widetilde{X}(T-t)$ for $t' < t \le t'''$.

Furthermore, there would exist a t^*, $t' < t^* \le T$, such that $X(t^*) = \widetilde{X}$, otherwise $X(t) < \widetilde{X}$ and, thus, $\dot{\psi}(t) = 2X(t) < 2\widetilde{X}$ and $\psi(t) < -2\widetilde{X}(T-t)$ for $t' < t \le T$. This implies $\psi(T) < 0$, which contradicts the assumption that $\psi(T) = 0$. Since $X(t^*) = \widetilde{X}$ and $X(t''') < \widetilde{X}$, there would be a t_1, $t''' < t_1 \le t^*$ such that $X(t_1) = \widetilde{X}$ and $X(t) < \widetilde{X}$ for $t''' \le t < t_1$. Therefore, $\dot{\psi}(t) = 2X(t) < 2\widetilde{X}$ and, thus, $\psi(t) < -2\widetilde{X}(T-t)$, $u(t) \le (D/P)$, and finally $\dot{X}(t) < 0$, for $t''' < t < t_1$. Since $X(t''') < \widetilde{X}$, we would have $X(t_1) < \widetilde{X}$. However, this contradicts the assumption that $X(t_1) = \widetilde{X}$. Therefore, we must have $\psi(t') = -2\widetilde{X}(T-t')$.

We now show that $X(t) = \widetilde{X}$ and $\psi(t) = -2\widetilde{X}(T-t)$ for $t' \le t \le T$ by contradiction. Assume there existed some α_1 and α_2, $t' < \alpha_1 < \alpha_2 < T$ such that $X(t) = \widetilde{X}$ for $t' \le t \le \alpha_1$, and $X(t) \ne \widetilde{X}$ for $\alpha_1 < t \le \alpha_2$. This would mean that $\dot{X}(t) \ne 0$ at $t = \alpha_1$. However, $X(t) = \widetilde{X}$ for $t' \le t \le \alpha_1$ and $\psi(t') = -2\widetilde{X}(T-t')$ should result in $\dot{\psi}(t) = 2\widetilde{X}$, $\psi(t) = -2\widetilde{X}(T-t)$, $u(t) = D/P$ and, thus, $\dot{X}(t) = 0$ for $t' \le t \le \alpha_1$ which contradicts $\dot{X}(t) \ne 0$ at $t = \alpha_1$. Therefore, we must have $X(t) = \widetilde{X}$ and $\psi(t) = -2\widetilde{X}(T-t)$ for $t' \le t \le T$. ■

Theorem 1: Let $\hat{X} > X(0) \ge X^* = -\beta D/P^2$ and A, B, t_Ψ, t_O satisfy the following equations:

$$AI_0\left(2\sqrt{C(T-t_\psi)}\right) + BK_0\left(2\sqrt{C(T-t_\psi)}\right) = \frac{2\left(X(0) - Dt_\psi - X^*\right)}{\sqrt{C}} \tag{15}$$

$$AI_1\left(2\sqrt{C(T-t_O)}\right) + BK_1\left(2\sqrt{C(T-t_O)}\right) = 0 \tag{16}$$

$$AI_1\left(2\sqrt{C(T-t_\psi)}\right) + BK_1\left(2\sqrt{C(T-t_\psi)}\right) = 2X^*\sqrt{T-t_\psi} \tag{17}$$

$$X^* = X(0) - Dt_\psi + \frac{P^2}{2\beta} \times\left[\frac{A}{\sqrt{C}}\left(I_0\left(2\sqrt{C(T-t_\psi)}\right) - I_0\left(2\sqrt{C(T-t_O)}\right)\right) - \frac{B}{\sqrt{C}}\left(K_0\left(2\sqrt{C(T-t_\psi)}\right) - K_0\left(2\sqrt{C(T-t_O)}\right)\right)\right], \tag{18}$$

$$\hat{X} = X^* + Dt_\psi \frac{I_0(2\sqrt{CT})}{I_0(2\sqrt{CT}) + I_0(\sqrt{C(T-t_\psi)}) - 2} \tag{19}$$

where $C = P^2/\beta$. Define (20), as shown at the bottom of the page, where $I_n(z)$ is the Modified Bessel function of the first kind of order n and $K_n(z)$ is the Bessel function of the second kind of order n (Neumann function).

$$\psi_X(t) = \begin{cases} \sqrt{T-t_\psi}\cdot\left[AI_1\left(2\sqrt{C(T-t_\psi)}\right) + BK_1\left(2\sqrt{C(T-t_\psi)}\right)\right] - 2X^*(T-t_\psi) - 2X(0)(t_\psi - t) + D\left(t_\psi^2 - t^2\right), 0 \le t < t_\psi \\ \sqrt{T-t}\cdot\left[AI_1\left(2\sqrt{C(T-t)}\right) + BK_1\left(2\sqrt{C(T-t)}\right)\right] - 2X^*(T-t), t_\psi \le t < t_O \\ -2X^*(T-t), t_O \le t \le T \end{cases} \tag{20}$$

Then

$$u(t) = \begin{cases} 0, & 0 \le t < t_\psi, \\ \frac{P\psi_X(t)}{2\beta(T-t)}, & t_\psi \le t \le T \end{cases} \tag{21}$$

is optimal.

Proof: In order to show the optimality of $u(t)$, we need to prove that

i) $\dot{\psi}_X(t) = 2X_E(t)$ and $\psi_X(T) = 0$;
ii) $u(t)$ is feasible;
iii) $u(t)$ and $\psi_X(t)$ maximize the Hamiltonian (11).

First, note that according to (8) and (21), $X_E(t) = X(0) - Dt$ for $0 \le t < t_\psi$. Then, from (20) we find

$$\dot{\psi}_X(t) = 2(X(0) - Dt) = 2X_E(t), \text{ for } 0 \le t < t_\psi. \tag{22}$$

Next, $\psi_X(t), t_\psi \le t < t_O$ satisfies the following differential equation [3]:

$$\ddot{\psi}_X(t) - P^2 \frac{\psi_X(t)}{\beta(T-t)} = -2D \tag{23}$$

which with respect to (8) and (21) can be rewritten as

$$\ddot{\psi}_X(t) = 2Pu(t) - 2D = 2\dot{X}_E(t) \text{ for } t_\psi \le t < t_O. \tag{24}$$

Thus, from (22) and (24), we have

$$\dot{\psi}_X(t) = 2X_E(t) \text{ for } 0 \le t < t_O.$$

For $t_O \le t \le T$, substituting (20) into (21) leads to $u(t) = D/P$. Thus, $Pu(t) - D = \dot{X}_E(t) = 0$, which results in $X_E(t) = X^*$ for $t_O \le t \le T$. Differentiating (20), we show that $2X_E(t) = 2X^* = \dot{\psi}_X(t)$. Finally, it is easy to verify that $\psi_X(T) = 0$. Therefore, i) is proven.

Let us now show $u(t)$ is feasible, that is, $0 \le u(t) \le U$. First, it can be shown that $X_E(t_O) = X^*$ [(18)–(20)], $\psi_X(t_O) = -2X^*(T - t_O)$ [(16) and (20)], as well as, $\psi_X(t)$, $X_E(t)$ and $u(t)$ satisfy the remaining conditions of Lemma 3. According to that lemma, $\psi_X(t) \le -2X^*(T-t)$ for $0 \le t \le t_O$. Thus, $\psi_X(t) \le -2X^*(T-t)$ and therefore $u(t) \le (D/P) < U$ for $0 \le t \le T$, which yields $\dot{X}_E(t) \le 0$, $0 \le t \le T$. Assume $X_E(0) > 0$ (in fact, this is ensured by the existence of t_Ψ). Since $X_E(t)$ is nonincreasing and $X^* < 0$, there must be a $t_X < t_O$, such that $X_E(t_X) = 0$. Therefore, $X_E(t) \le 0$ and, thus, $\dot{\psi}_X(t) = 2X_E(t) \le 0$, $t_X \le t \le T$ and $\dot{\psi}_X(t) > 0$, $0 \le t < t_X$. Considering $\psi_X(T) = 0$, we have $\psi_X(t) \ge 0$, $t_X \le t \le T$. Thus $t_\Psi < t_X$. Also $\psi_X(t) < 0$, $0 \le t < t_\Psi$, and $\psi_X(t) \ge 0$, $t_\Psi \le t \le T$. Taking (21) into account, we conclude that $0 \le u(t)$ for $0 \le t \le T$. Combining this with the fact $u(t) \le (D/P) < U$ for $0 \le t \le T$ that we have just proven, we conclude that $u(t)$ is feasible.

Finally, it is easy to observe, that $u(t)$ and $\psi_X(t)$ determined by (20) and (21) maximize the Hamiltonian (11). ■

V. A Description of the Optimal Control

The optimal control when $X(0) \ge X^*$ is dependent upon the initial inventory $X(0)$ in the following manner.

Case 1) $X(0) \ge DT$, $u^*(t) = 0$, $0 \le t \le T$. This is the first special case in the last section. Only the first phase of the three-phase control is used.

Case 2) $DT > X(0) \ge \hat{X}$. The optimal control is defined as shown in (25) and (26) at the bottom of the page, and A and t_Ψ are obtained by solving the following equation:

$$\begin{aligned} &AI_1\left(2\sqrt{C(T-t_\psi)}\right) - 2X^*\sqrt{T-t_\psi} = 0, \\ &AI_0\left(2\sqrt{C(T-t_\psi)}\right) = \frac{2(X(0) - Dt_\psi - X^*)}{\sqrt{C}}. \end{aligned} \tag{27}$$

This case has the first two phases described in Theorem 1: Initially $u^*(t) = 0$ when $t < t_\Psi$ and then $u^*(t)$ becomes positive, but still small enough so that the average inventory level $X_E(t)$ continues declining. Since the initial inventory is relatively large, $X_E(t)$ will never reach the critical value X^* and thus the third phase of the control will not be entered.

Case 3) $\hat{X} > X(0) \ge X^*$. We have the second special case determined by Theorem 1 with a three-phase control, which is illustrated in Fig. 1(a).

Note that the proof of Case 2) is very similar to that of Theorem 1 and thus omitted.

VI. Conclusion

An optimal offline control scheme is developed in this note for a production system with random yield and constant demand. Depending upon the initial inventory level, the optimal control may have up to three phases. In the first phase, the optimal production rate is either at its maximum or its minimum, like in the traditional threshold control. The optimal production rate in the second phase is determined by a set of complex nonlinear equations containing Bessel functions. In the third phase, similar to the traditional threshold control, the system enters a steady-state characterized by a constant production rate inversely proportional to the expected yield.

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$$\psi_X(t) = \begin{cases} A\sqrt{T-t_\psi}\cdot I_1\left(2\sqrt{C(T-t_\psi)}\right) + \frac{2D}{C}(T-t_\psi) - 2X(0)(t_\psi - t) + D\left(t_\psi^2 - t^2\right), & 0 \le t < t_\psi \\ A\sqrt{T-t}\cdot I_1\left(2\sqrt{C(T-t)}\right) + \frac{2D}{C}(T-t), & t_\psi \le t \le T \end{cases} \tag{25}$$

$$u^*(t) = \begin{cases} 0, & 0 \le t < t_\Psi, \\ \frac{P\psi_X(t)}{2\beta(T-t)}, & t_\Psi \le t \le T \end{cases} \tag{26}$$

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Task-Space Adaptive Setpoint Control for Robots With Uncertain Kinematics and Actuator Model

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***Abstract*—In this note, the adaptive setpoint control problem of robotic manipulators in the presence of uncertainties in both kinematics and actuator model is addressed. Two new task-space control methods are proposed to overcome the uncertainties. Sufficient conditions for choosing the feedback gains, estimated Jacobian matrix and estimated actuator model are provided to guarantee system stability. Experimental results are presented to verify the practical feasibility of the proposed control methods.**

I. INTRODUCTION

A great many control schemes for robotic manipulators have been developed in literature during the past few decades. In many of these control methods [1]–[6], the controller is designed in joint space. Since for most applications of robotic manipulators the desired position or path is specified in task space, one principle limitation associated with these joint-space control methods is that the desired joint position or path must be obtained by solving the inverse kinematics. To avoid the problem of solving inverse kinematics, Takegaki and Arimoto [7] proposed a task-space controller for setpoint control in Cartesian space using a transposed Jacobian matrix. Many other task-space control schemes are proposed later [8]–[11]. To apply these task-space control schemes, exact knowledge of the Jacobian matrix from joint space to task space is required. If uncertainties exist in the kinematics, these controllers [1]–[11] may give degraded performance and may incur instability. To deal with the problem of uncertain kinematics, Cheah *et al.* [12]–[14] proposed several task-space feedback laws with uncertain kinematics from joint space to task space.

However, most control methods proposed in literature, including the control methods previously mentioned [1]–[14], are designed at the torque input level and the actuator part is neglected. As shown by Good *et al.* [15], the actuator model constitutes an important part of the complete robot system and may cause detrimental effects when neglected in the design procedure. Recently, actuator dynamics has been explicitly included in control schemes and some research work has been devoted to deal with this problem as can be found in [16]–[25]. However most of these control schemes are designed in joint space and exact kinematics information is assumed to be known. To our knowledge, no result has been proposed for task-space setpoint control with the presence of uncertainties in both kinematics and actuator model. The main theoretical challenge of this control problem is to guarantee the stability of closed-loop system in the presence of both kinds of uncertainties without invoking overparameterization method which is often used to deal with multiple uncertainties in system. In this note, we propose a new regressor construction concept using online updating information of the adaptive parameters. Based on the novel adaptive regressor proposed, two task-space controllers are developed which can deal with the uncertainties in kinematics and actuator model at the same time. The proposed schemes do not need accurate information about the actuator model, dynamics and kinematics of the robot system except the gravity regressor matrix. And compared with the existing results in literature, the dimension of the closed loop system is lower and the controller structure can be greatly simplified especially when degree of freedom of the robot increases since the two kinds of uncertainties are dealt with separately and simultaneously through the novel regressor proposed. Sufficient conditions revealing the coupling effects of uncertain actuator model and kinematics on system stability are provided and are illustrated in a three-dimensional (3-D) figure. It is shown by means of experiments that the proposed controllers are practically applicable and systematic tuning procedure of the control gains is provided.

II. ROBOT KINEMATICS AND DYNAMICS

In order to describe a task for the robot manipulator, the desired path for the end effector is usually specified in task space. Let $X \in R^m$ represents the position vector of the manipulator in task space defined by [9], [12]

$$X = h(q) \tag{1}$$

where $q \in R^n$ is a vector of generalized joint coordinates, $h(\cdot) \in R^n \rightarrow R^m$ $(m \leq n)$ is generally a nonlinear transformation describing the relation between the joint and task space. The velocity vector $\dot{X}$ is therefore related to $\dot{q}$ as

$$\dot{X} = J(q)\dot{q} \tag{2}$$

where $J(q) \in R^{m \times n}$ is the Jacobian matrix of mapping from joint space to task space. Note that if the robot's kinematics is uncertain, the Jacobian matrix becomes uncertain too.

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