Supply Chain With Inventory Review and Dependent Demand Distributions: Dynamic Inventory Outsourcing

Konstantin Kogan, Sheldon Lou, Charles S. Tapiero, and Matan Shnaiderman

***Abstract*—In this paper, we consider inventory outsourcing by a producer to a distributor. The distributor charges a cost for each unit it handles and the manufacturer responds with a production and inventory policy over a finite contract period. As a result, the two parties enter a noncooperative differential game. We address the effect of information asymmetry in such a game under a stochastic demand when the inventory level can only be observed by the manufacturer intermittently.**

***Note to Practitioners*—We demonstrate that even for the seemingly simple one-part-type system with a single inventory review, the inclusion of the random demand still leads to a nontrivial optimal production control for the manufacturer. As to the distributor, his charge for handling manufacturer's inventories is determined by his leadership in the supply chain and depends on the remaining number of periods to go. While the desire to charge as much as possible for handling inventories at the last period prevails, the distributor reduces the charge when there are two periods to go.**

***Index Terms*—Differential games, dynamic programming/optimal control applications, inventory/production policies.**

I. Introduction

FIRMS within a supply chain often pursue their own and mutually conflicting objectives, leading to intra-supply-chain competitions and the deterioration of the overall performance. The effect of such competitions on supply chains and their performance is well studied under a static framework. Extensive reviews focusing on such competition-related aspects include discussions on integrated inventory models [11], game theory in supply chains [2], price quantity discounts [21], and competition and coordination [18]. However, contemporary business conditions are often characterized by highly volatile situations changing in a stochastic manner. As a result, decisions and adjustments need to be made frequently based on system state updates. Such a dynamic nature of supply chains exceeds the scope of the static framework and thus demands further study.

Our paper focuses on dynamic behaviors of various parties in a supply chain. Although differential game seems an instinctive choice for such an analysis, due to its mathematical difficulties, in particular, when decisions have to be made continuously, the supply chain management literature has primarily been concerned with **deterministic, but not stochastic**, differential models [2]. For instance, Jorgenson [10] derives an open-loop Nash equilibrium for a channel with static deterministic demand. Eliashberg and Steinberg [6] use the Stackelberg solution approach in a game involving a manufacturer and a distributor (both of unlimited capacity) and a quadratic seasonal demand. Assuming a constant wholesale price with which the manufacturer charges the distributor and no backlog allowed, they investigate the impact of a deterministic seasonal pattern upon various policies of the channel. To address the deterministic seasonal demands, Desai [3] suggests a numerical analysis of the Stackelberg solution under unlimited production capacity. For additional applications of differential games in management science and operations research, we refer to reviews by Feichtinger and Jorgenson [9], Kogan and Tapiero [16], and He *et al.* [12].

In this paper, we analyze a differential inventory game with a **stochastic demand** in a supply chain employing the so-called inventory outsourcing, which is characterized by a distributor who holds the manufacturer's inventory and charges him a cost. In fact, it is a special case of a broader technique called Vendor Managed Inventory (VMI), which has been getting increasingly popular among large producers and distributors in a supply chain (see, for example, [4], [17], [20], and [23] for deliberations on information sharing between parties employing VMI). Such inventory management services have grown steadily over the last half century and now facilitate warehouse, logistics, delivery, courier, and storage distribution. Today distributors offer comprehensive and integrated menus of services to retailers, manufacturers and various agencies in a supply chain covering diverse areas such as automotive, healthcare, medical, supply and fixed asset management.

The contractual mode of the inventory outsourcing stated above portrays a supply chain with two players engaging in a differential game. In particular, we consider the situation when the distributor is the leader who sets the charge for carrying inventory and the manufacturer, a follower who responds to

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the charge with a production policy. This depicts a sequential decision making scheme (called a Stackelberg strategy).

The contribution of our framework includes the following.

i) Illustrating the effect of periodic VMI dynamics on a continuous-time supply chain model.
ii) Analyzing a stochastic differential game and providing closed form solutions.
iii) Determining the effect of leadership on supply chain decisions over time in terms of both production policies and inventory charges.

Based on these features, we provide a number of insights, which extend the previous results for static models. Specifically, we find that the competition forces the distributor to increase his charges for handling inventories. Note, that the situation is very different in a centralized supply chain, where inventory charges, considered internal transfers, are typically marginal. We show that this phenomenon corresponds to the so-called double marginalization effect well-studied in static settings. We demonstrate that this effect depends on the leadership in the supply chain and on the remaining number of periods to go. Thereby, it is affected by demand distributions. Moreover, the dynamic setting shows that increased length of inventory review period impacts both production and inventory dynamics of the manufacturer and the inventory handling charges of the distributor. Consequently, the implied double-marginalization worsens.

II. Problem Formulation

We assume that the distributor's goal is to minimize its expected cost (or maximize its profit)

$$I = E\left[\int_0^T (m - c^+(t))X^+(t)dt\right] \rightarrow \min_{c^+} \tag{1}$$

$$\text{s.t. } m \le c^+(t) \le s \tag{2}$$

where m is the distributor's marginal cost, s is the maximum willingness to pay for the inventory service, $X(t)$ is the inventory $(X^+(t) = \max\{X(t), 0\})$ or backlog $(X^-(t) = \max\{-X(t), 0\})$, and $c^+(t)$ is the cost the distributor charges the manufacturer for holding a unit of product. It is also the control variable for the distributor.

Most theoretical work related to stochastic, continuous-time production with periodic review is focused on single-period models (e.g., Kogan *et al.* [15]. Our production model introduces an update before the end of production period, i.e., it considers a two-period review approach. As defined below, the model is inspired by business cases in which manufacturers produce products with relatively short life cycles. In such cases, a single inventory update at a predetermined point may suffice to identify the demand for the remaining part of the production horizon. For example, Fisher *et al.* [7], [8] report examples in the apparel industry, where highly accurate demand forecasts are made after observing only 11% to 20% of the total demand.

Specifically, the inventory dynamics of the manufacturer is defined as

$$\dot{X}(t) = u(t) - d_i(t) \tag{3}$$

where $X(0)$ is a known initial inventory level and $u(t)$, the production rate at time t. The production rate is bounded by the capacity U, the maximum amount of products that the manufacturer can produce per time unit

$$0 \le u(t) \le U. \tag{4}$$

In the above equation, $d_i(t)$, $i = 1$ for $0 \le t < \tau$ and $i = 2$ for $\tau \le t < T$, is the demand rate. Similar to many production control studies (see, for example, [13], [19], [14]), we assume that the demand rate is exogenous and constant at each period. This implies that the demand does not depend on the level of available stocks and the price (the price elasticity of demand is zero). Such low-price elasticity is typical when substitute products are scarce or the necessity of the product is high. Typically, products requiring a small portion of the customer's income tend to have lower elasticity. (We plan to address the cases when demand varies in response to item availability, i.e., inventory level, and to item price in our future research.) Specifically, we assume that $d_1(t) = d_1$ and $d_2(t) = d_2$, where d_1 and d_2 are random variables whose values are unknown in $0 \le t < \tau$ but known at $t = \tau$. The probability density function $f_1(D_1)$ of d_1 and the conditional distribution function $f_2(D_2|D_1)$ are known. Also, $f_i(\cdot)$ and the corresponding cumulative functions, $F_i(\cdot) i = 1, 2$, are continuously differentiable. To simplify the expressions, we further assume that $0 \le d_i \le U$.

The manufacturer's goal is to minimize expected inventory costs

$$J = E\left[\int_0^T C(X(t))dt\right] \rightarrow \min_u \tag{5}$$

where

$$C(X(t)) = c^+(t)\max\{X(t), 0\} + c^- \max\{-X(t), 0\}. \tag{6}$$

In the above expression, c^- is the unit backlog (shortage) cost. Henceforth, we also assume that the inventory unit cost $c^+(t)$ may be adjusted once at the inventory review point $t = \tau$, i.e., $c^+(t) = \begin{cases} c_1^+, & \text{if } 0 \le t < \tau \\ c_2^+, & \text{if } \tau \le t \le T \end{cases}$.

We first analyze the deterministic part of the problem.

III. The Deterministic Component of the Problem

Consider time interval $[\tau, T]$. At this interval problem, (1)–(4) takes the following deterministic form:

$$I_{\det} = \int_\tau^T (m - c_2^+)X^+(t)dt \rightarrow \min \tag{7}$$

s.t. to (2)–(4).

The optimal solution (commonly referred in game theory as the best response function) for this problem is straightforward because dynamic (3) does not depend on $c^+(t)$ explicitly. Thus, the optimal solution for the distributor is a trivial one: charge the producer at the highest possible value, $c_2^+ = s$ when $X(t) > 0$. This charge does not affect the objective function when $X(t) = 0$ as there is nothing to store. This corresponds to the severe double marginalization, as the distributor levies the maximum charge for each unit stored. The optimal production policy for the producer is also trivial. It depends on the initial inventory $X(\tau)$ as the following: If $X(\tau)$ is larger than zero, stop the production until the inventory reaches zero; then maintain a production rate equal to the demand rate to keep inventory at zero. On the other hand, as long as $X(\tau)$ is less than zero, produce at the maximum rate U. The optimal production rate is thus a function of $X(\tau)$ and t, denoted $\pi^*(X(\tau),t)$ and is summarized in the following theorem.

Theorem 1: For $\tau \le t \le T$, given inventory level $X(\tau)$, the Stackelberg equilibrium is described by the unique optimal production policy $u(t)=\pi^*(X(\tau),t)$, see the first equation at the bottom of the page and the optimal inventory holding charge $c_2^+ = s$ when $X(t)>0$, otherwise $c_2^+\in[m,s]$. ∎

The related objective function values for the distributor can be calculated as the following:

$$\begin{aligned}
&I_{\det} = 0 \text{ if } X(\tau)\le 0;\\
&I_{\det} = (m-c_2^+)\left(X(\tau)(T-\tau)-\frac{D_2}{2}(T-\tau)^2\right),\\
&\quad\text{if } X(\tau) > D_2(T-\tau)\\
&\text{and } I_{\det} = (m-c_2^+)\frac{X(\tau)^2}{2D_2}\\
&\quad\text{if } 0 < X(\tau)\le D_2(T-\tau).
\end{aligned}\tag{8}$$

The objective functions for the producer are similar

$$\begin{aligned}
J_{\det} &= -c^-\left(-\frac{X(\tau)^2}{U-D_2}+\frac{U-D_2}{2}\left(\frac{X(\tau)}{U-D_2}\right)^2\right)\\
&= c^-\frac{X(\tau)^2}{2(U-D_2)}\\
&\text{for } D_2 < U+\frac{X(\tau)}{T-\tau}
\end{aligned}$$

otherwise

$$\begin{aligned}
J_{\det} &= -\int_\tau^T c^-(X(\tau)+(U-D_2)(t-\tau))dt\\
&= -c^-\left(X(\tau)(T-\tau)+\frac{U-D_2}{2}(T-\tau)^2\right).
\end{aligned}$$

IV. The Stochastic Component of the Problem

Let us now consider the optimal solution in $[0,\tau]$. Applying conditional expectation to the objective function (1) and accounting for (8), we obtain equation (9) at the bottom of the page, and the last equation at the bottom of the page.

$$\pi^*(X(\tau),t)=\begin{cases}
0, & \text{if } X(\tau)>D_2(T-\tau), \tau\le t\le T\\
0, & \text{if } 0\le X(\tau)\le D_2(T-\tau), \tau\le t<\tau+\frac{X(\tau)}{D_2}\\
D_2, & \text{if } 0\le X(\tau)\le D_2(T-\tau), \tau+\frac{X(\tau)}{D_2}\le t\le T\\
D_2, & \text{if } (D_2-U)(T-\tau)<X(\tau)\le 0, \tau-\frac{X(\tau)}{U-D_2}\le t\le T\\
U, & \text{if } (D_2-U)(T-\tau)<X(\tau)\le 0, \tau\le t<\tau-\frac{X(\tau)}{U-D_2}\\
U, & \text{if } X(\tau)\le (D_2-U)(T-\tau), \tau\le t\le T
\end{cases}$$

$$\begin{aligned}
I = &\int_0^\tau\int_0^{\left(X(0)+\int_0^t u(s)ds/t\right)}(m-c_1^+)\left(X(0)+\int_0^t u(s)ds-D_1t\right)\\
&\times f_1(D_1)dD_1dt + E\left[\int_\tau^T (m-c_2^+)X^+(t)dt\right]
\end{aligned}\tag{9}$$

$$\begin{aligned}
&E\left[\int_\tau^T (m-c_2^+)X^+(t)dt\right]\\
&\int_0^{X(0)+\int_0^\tau u(s)ds/\tau} f_1(D_1)(m-c_2^+)\\
&\times\left[\int_0^{(X(\tau)/T-\tau)}\left(X(\tau)(T-\tau)-\frac{D_2}{2}(T-\tau)^2\right)f_2(D_2|D_1)dD_2+\int_{(X(\tau)/T-\tau)}^U \frac{X(\tau)^2}{2D_2}f(D_2|D_1)dD_2\right]dD_1
\end{aligned}$$

Then, denoting $Y(\tau) = X(0) + \int_0^\tau u(s)ds$, objective function (9) transforms into

$$I = \int_0^\tau \int_0^{(Y(t)/t)} (m - c_1^+)(Y(t) - D_1 t) \times f_1(D_1)dD_1 dt + \eta(Y(\tau)) \to \min_{c_1^+}, \tag{10}$$

and equation (11) at the bottom of the page. Thus, we have transferred the original stochastic problem into a deterministic one.

A. Optimal Response of the Manufacturer

We start the analysis by discussing the optimal production strategy for the producer in response to a c_1^+, $m \le c_1^+ \le s$. Consider the objective function (5) for the producer

$$J = E\left[\int_0^T C(X(t))dt\right] = E\left[\int_0^\tau C(X(t))dt\right] + E\left[\int_\tau^T C(X(t))dt\right]. \tag{12}$$

Applying conditional expectation and (3) and (6), we have equation (13) shown at the bottom of the page.

As we have explained, the last term in (13), denoted J_{det}, is deterministic for given $X(\tau)$ and D_2 (i.e., at $t = \tau$). We employ Theorem 1 and consider two cases, $X(\tau) > 0$ and $X(\tau) \le 0$.

1) $X(\tau) > 0$. Since $X(\tau) = X(0) + \int_0^\tau u(s)ds - d_1\tau$, the probability of $X(\tau)$ being larger than 0 is equal to $\int_0^{\left(X(0)+\int_0^\tau u(s)ds/\tau\right)} f_1(D_1)dD_1$.
Therefore,
$E_{X(\tau)>0}\left[\int_\tau^T C(X(t))dt\right] = \int_0^{\left(X(0)+\int_0^\tau u(s)ds/\tau\right)} f_1(D_1)\left[\int_0^{(X(\tau)/T-\tau)} c_2^+(X(\tau)(T-\tau) - (D_2/2)(T-\tau)^2) f_2(D_2|D_1)dD_2 + \int_{(X(\tau)/T-\tau)}^U c_2^+(X(\tau)^2/2D_2) f(D_2|D_1)dD_2\right] dD_1$.
2) $X(\tau) \le 0$. Similarly, $E_{X(\tau)\le 0}\left[\int_\tau^T C(X(t))dt\right] = \int_{(X(0)+\int_0^\tau u(s)ds/\tau)}^U f_1(D_1)\left[\int_0^{U+(X(\tau)/T-\tau)} c^-\left(X(\tau)^2/2(U-D_2)\right) f_2(D_2|D_1)dD_2 - \int_{U+(X(\tau)/T-\tau)}^U c^-(X(\tau)(T-\tau) + (U-D_2/2)(T-\tau)^2) f_2(D_2|D_1)dD_2\right] dD_1$.

Since $E\left[\int_\tau^T C(X(t))dt\right] = E_{X(\tau)>0}\left[\int_\tau^T C(X(t))dt\right] + E_{X(\tau)\le 0}\left[\int_\tau^T C(X(t))dt\right]$, the objective function (13) at the bottom of the page takes the following form of equation (14) at the bottom of the page and see equation (15) at the bottom of the next page.

Problem (4), (14), and (15) are a canonical deterministic optimal control problem which can be studied with the aid of the maximum principle. Since all constraints are linear, the maximum principle-based optimality conditions are not only necessary but also sufficient provided that the objective func-

$$\eta(Y(\tau)) = \int_0^{(Y(\tau)/\tau)} f_1(D_1)(m - c_2^+) \times \left[\int_0^{(Y(\tau)-D_1\tau/T-\tau)} \left((Y(\tau) - D_1\tau)(T-\tau) - \frac{D_2}{2}(T-\tau)^2\right) f_2(D_2|D_1)dD_2 + \int_{(Y(\tau)-D_1\tau/T-\tau)}^U \frac{(Y(\tau)-D_1\tau)^2}{2D_2} f_2(D_2|D_1)dD_2\right] dD_1. \tag{11}$$

$$J = \int_0^\tau \left\{ \int_0^{\left(X(0)+\int_0^t u(s)ds/t\right)} c^+ \left(X(0) + \int_0^t u(s)ds - D_1 t\right) \times f_1(D_1)dD_1 - \int_{\left(X(0)+\int_0^t u(s)ds/t\right)}^U c^-\left(X(0) + \int_0^t u(s)ds - D_1 t\right) f_1(D_1)dD_1 \right\} dt + E\left[\int_\tau^T C(X(t))dt\right]. \tag{13}$$

$$J = \int_0^\tau \left\{ \int_0^{(Y(t)/t)} c_1^+ (Y(t) - D_1 t) f_1(D_1)dD_1 - \int_{(Y(t)/t)}^U c^- (Y(t) - D_1 t) f_1(D_1)dD_1 \right\} dt + \varphi(Y(\tau)) \to \min \tag{14}$$

tion (14) is convex. Moreover, if the latter is strictly convex (if $(\partial F_1(D)/\partial D) > 0$ and $(\partial^2\varphi(Y(\tau))/\partial Y^2) > 0$), this problem has a unique solution. We thus construct the Hamiltonian as

$$H = -\int_0^{(Y(t)/t)} c_1^+ \left(Y(t) - D_1 t\right) f(D_1) dD_1 + \int_{(Y(t)/t)}^{U} c^- \left(Y(t) - D_1 t\right) f(D_1) dD_1 + \psi(t)u(t) \tag{16}$$

where the co-state variable $\psi(t)$ is determined by the co-state differential equation

$$\dot{\psi}(t) = -\frac{\partial H}{\partial Y(t)} = (c_1^+ + c^-)F_1\left(\frac{Y(t)}{t}\right) - c^- \tag{17}$$

with boundary (transversality) condition

$$\psi(T) = -\frac{\partial\varphi(Y(\tau))}{\partial Y(\tau)}. \tag{18}$$

According to the maximum principle, the optimal control that maximizes the Hamiltonian is

$$u(t) = \begin{cases} U, & \text{if } \psi(t) > 0 \\ 0, & \text{if } \psi(t) < 0 \\ b \in [0, U], & \text{if } \psi(t) = 0 \end{cases}. \tag{19}$$

1) Solution: First, we resolve the ambiguity of the third condition from (19). This is accomplished in the following lemma by differentiating the condition $\psi(t) = 0$ and taking into account (17), which results in

$$F_1\left(\frac{Y(t)}{t}\right) = \frac{c^-}{c_1^+ + c^-}. \tag{20}$$

Lemma 1: Let $\psi(t) = 0$ at a measurable interval r and define $F_1(\beta) = (c^-/c_1^+ + c^-)$. If $\beta \le U$, then $Y(t) = \beta t$ and $u(t) = \beta$ for $t \in r$.

Proof: See the Appendix. ■

Note that (20) corresponds to the solution of the well-known news-vendor problem. Accordingly, the singular condition, $\psi(t) = 0$, which is most attractive in linear control systems, presents the case when the service level $(c^-/c_1^+ + c^-)$ can be ensured by an optimal production rate, $u(t) = \beta$, chosen so that the probability that the demand rate does not exceed the

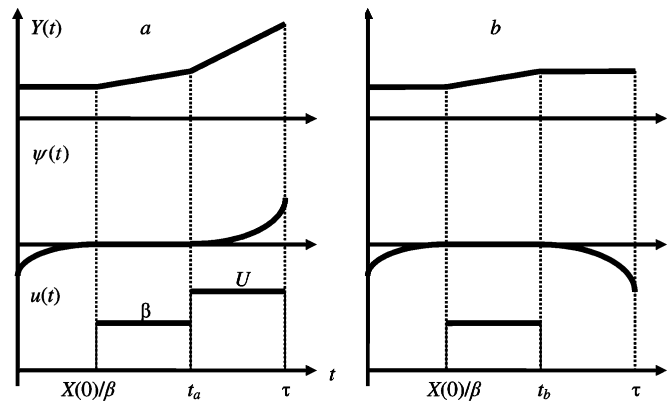

Fig. 1. Optimal control over the first period for $X(0) > 0$ when (a) $(\partial\varphi(Y(\tau))/\partial Y(\tau)) < 0$ and (b) $(\partial\varphi(Y(\tau))/\partial Y(\tau)) > 0$.

production rate β is $(c^-/c_1^+ + c^-)$. Equation (20) also shows that the greater the cost c_1^+ charged by the distributor, the lower the service level $(c^-/c_1^+ + c^-)$, the production rate $u(t) = \beta$ and thereby the inventory level. This corresponds to the double marginalization effect. That is, the competition causes the manufacturer to produce and stock less than that demanded by the system-wide optimal solution.

We next introduce two switching points t_a and t_b which satisfy

$$\int_{t_a}^{\tau} \left[(c_1^+ + c^-)F_1\left(\frac{\beta t_a + U(t - t_a)}{t}\right) - c^-\right] dt = -\frac{\partial\varphi(Y(\tau))}{\partial Y(\tau)}\bigg|_{Y(\tau)=\beta t_a + U(\tau - t_a)}, \tag{21}$$

$$\int_{t_b}^{\tau} \left[(c_1^+ + c^-)F_1\left(\frac{\beta t_b}{t}\right) - c^-\right] dt = -\frac{\partial\varphi(Y(\tau))}{\partial Y(\tau)}\bigg|_{Y(\tau)=\beta t_b} \tag{22}$$

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respectively, and assume that the production system is balanced, i.e., $\beta \le U$.

Consider first the case of a non-negative initial inventory level. Two most general cases are described in the following two lemmas and depicted in Fig. 1.

$$\begin{aligned}\varphi(Y(\tau)) = &\int_0^{(Y(\tau)/\tau)} f_1(D_1)\Bigg[\int_0^{(Y(\tau)-D_1\tau/T-\tau)} c_2^+ \left((Y(\tau) - D_1\tau)(T - \tau) - \frac{D_2}{2}(T-\tau)^2\right) f_2(D_2|\,D_1)dD_2 \\ &\qquad + \int_{(Y(\tau)-D_1\tau/T-\tau)}^{U} c_2^+ \frac{(Y(\tau) - D_1\tau)^2}{2D_2} f_2(D_2|\,D_1)dD_2\Bigg] dD_1 \\ &+ \int_{(Y(\tau)/\tau)}^{U} f(D_1)\Bigg[\int_0^{U+(Y(\tau)-D_1\tau/T-\tau)} c^- \frac{(Y(\tau) - D_1\tau)^2}{2(U - D_2)} f_2(D_2|\,D_1)dD_2 \\ &\qquad - \int_{U+(Y(\tau)-D_1\tau/T-\tau)}^{U} c^- \left((Y(\tau) - D_1\tau)(T-\tau) + \frac{U - D_2}{2}(T-\tau)^2\right) f_2(D_2|\,D_1)dD_2\Bigg] dD_1\end{aligned} \tag{15}$$

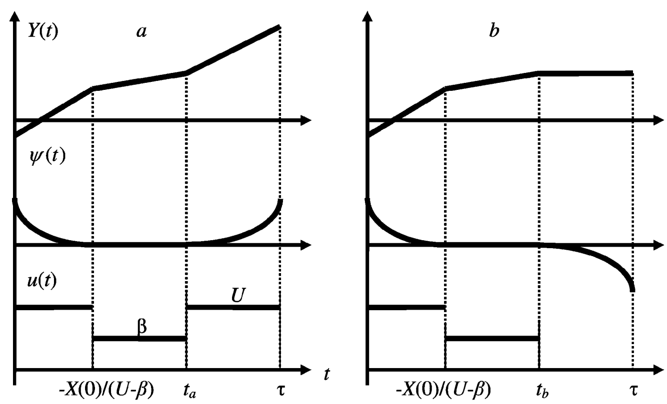

Fig. 2. Optimal control over the first period for $X(0) < 0$ when (a)$(\partial\varphi(Y(\tau))/\partial Y(\tau)) > 0$ and (b) $(\partial\varphi(Y(\tau))/\partial Y(\tau)) > 0$.

Lemma 2: Let $X(0) \geq 0$ and $X(0) < \beta\tau$. If $t_a > (X(0)/\beta)$ and $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=\beta t_a+U(\tau-t_a)} < 0$, then the optimal control is $u(t) = 0$ for $0 \leq t < (X(0)/\beta)$, $u(t) = \beta$ for $(X(0)/\beta) \leq t < t_a$, and $u(t) = U$ for $t_a \leq t \leq \tau$.

Proof: See the Appendix. ■

Note that the optimal control described in Lemma 2 consists of three different trajectories. If the feasibility requirement $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=\beta t_a+U(\tau-t_a)} < 0$ is not met, then instead of producing at the maximum rate the third trajectory implies no production at all. This is summarized in the following lemma.

Lemma 3: Let $X(0) \geq 0$ and $X(0) < \beta\tau$. If $t_b > (X(0)/\beta)$ and $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=\beta t_b} > 0$, then the optimal control is $u(t) = 0$ for $0 \leq t < (X(0)/\beta)$, $u(t) = \beta$ for $(X(0)/\beta) \leq t < t_b$, and $u(t) = 0$ for $t_b \leq t \leq \tau$. ■

Proofs for Lemmas 3 as well as for Lemmas 4–5 are similar to Lemma 2 and thus omitted. The two general cases when $X(0) < 0$ are shown in the following two lemmas (see Fig. 2).

Lemma 4: Let $X(0) < 0$ and $X(0) > -\tau(U-\beta)$. If $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=\beta t_a+U(\tau-t_a)} < 0$ and $t_a > (-X(0)/U-\beta)$, then the optimal control is $u(t) = U$ for $0 \leq t < (-X(0)/U-\beta)$, $u(t) = \beta$ for $(-X(0)/U-\beta) \leq t < t_a$, and $u(t) = U$ for $t_a \leq t \leq \tau$. ■

Lemma 5: Let $X(0) < 0$ and $X(0) > -\tau(U-\beta)$. If $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=\beta t_b} > 0$ and $t_b > (-X(0)/U-\beta)$, then the optimal control is $u(t) = U$ for $0 \leq t < (-X(0)/U-\beta)$, $u(t) = \beta$ for $(-X(0)/U-\beta) \leq t < t_b$, and $u(t) = 0$ for $t_b \leq t \leq \tau$. ■

Lemmas 2–5 along with Figs. 1 and 2 illustrate the effect of uncertainty on optimal production policies. Specifically, according to conditions (19) and Lemma 1, the production control under piecewise constant inventory costs involves only three optimal production regimes: $u(t) = 0$, $u(t) = \beta$, and $u(t) = U$. Lemmas 2–5 determine the sequencing of these regimes. The sequencing is due to the level of demand expectation. Unless the expectation is extremely high (so that the production at the maximum rate U is needed at all times), or extremely low (so that no production is needed), the most attractive regime is $u(t) = \beta$, as discussed above. This regime is followed by the maximum production rate if the demand expectation is high (but not extremely high), as observed from Figs. 1(a) and 2(a). Otherwise, if the expectation is low (but not extremely low), the singular regime is naturally followed by no production. The initial regime of a production period depends on initial inventories of the period. Indeed, if there is no inventory at the beginning of the period, then $u(t) = \beta$ from the beginning until the manufacturer has to switch to either maximum or minimum production level in anticipation of either high or low demand. This solution will be shown in Lemma 6 below and is a special case of that described in Lemmas 2–5. However, if the manufacturer initially has a surplus of inventory, then it is beneficial to get rid of the surplus first before entering the singular regime $u(t) = \beta$, as stated by Lemmas 2 for the case of high demand expectation and by Lemma 3 for low demand expectation (see Fig. 1(a) and (b), respectively). On the other hand, if there is a shortage at the beginning of the period, then it is optimal to start from production at maximum rate to eliminate the shortage as fast as possible, as stated in Lemmas 4 and 5 and shown in Fig. 2.

B. Stackelberg Equilibrium With the Distributor Being the Leader

When the distributor is the leader the optimal production policy defined in Lemmas 2–5 is substituted into the distributor's problem. Lemmas 2–5 identify four general types of optimal solutions and a number of suboptimal cases whose solutions are special cases of a generally optimal policy. Each of these cases thus induces a corresponding equilibrium. To avoid massive mathematical expressions we here focus only on two cases both of which are based on the common assumption that initial inventory level is zero, $X(0) = 0$. Then, the first switching point in Lemmas 2–5 vanishes and the optimal solution (the best response) takes the following form.

Lemma 6: The optimal production policy is the following.

- *Low demand expectation*: if $c^-\tau \leq (\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=0}$, then $u(t) = 0$ for $0 \leq t \leq \tau$, if $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=0} < c^-\tau$ but $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=\beta\tau} > 0$, then $u(t) = \beta$ for $0 \leq t < t_b$ and $u(t) = 0$ for $t_b \leq t \leq \tau$.
- *High demand expectation*: if $((c_1^+ + c^-)F(U) - c^-)\tau > -(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=U\tau}$ but $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=\beta\tau} < 0$, then $u(t) = \beta$ for $0 \leq t < t_a$ and $u(t) = U$ for $t_a \leq t \leq \tau$, otherwise if $((c_1^+ + c^-)F(U) - c^-)\tau \leq -(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=U\tau}$, then $u(t) = U$ for $0 \leq t \leq \tau$. ■

To find Stackelberg equilibrium, we substitute variable $Y(t)$ along with the first production policy from Lemma 6 (induced by low demand expectation of the supply chain) into (10) and (11). This converts the dynamic problem into a static one, to which we can apply the first order optimality condition

$$\frac{\partial I}{\partial c_1^+} = 0. \tag{23}$$

Let us denote a solution of (23) in c_1^+ as γ. We thus have proved the following theorem.

Theorem 2: Assume $X(0) = 0$, $(\partial F_1(D)/\partial D) > 0$, $(\partial^2\varphi(\beta t_b)/\partial Y^2) > 0$ and the distributor is the leader in the supply chain. If $(\partial^2 I/\partial c_1^{+2}) > 0$, $(\partial\varphi(0)/\partial Y) < c^-\tau$, $(\partial\varphi(\beta\tau)/\partial Y) > 0$ and $m \leq \gamma \leq M$, then $u(t) = \beta$ for

$0 \le t < t_b$, $u(t) = 0$ for $t_b \le t \le \tau$, and $c_1^+ = \gamma$ constitute a unique Stackelberg equilibrium in the differential inventory outsourcing game for $0 \le t \le \tau$. ■

Similarly, we can determine the equilibrium for the case of high demand expectation by substituting the production policy from Lemma 6 into (10) and (11). Denoting the solution to (23) for such a case as ρ, we conclude with the following theorem.

Theorem 3: Assume $X(0) = 0$, $(\partial F_1(D)/\partial D) > 0$, $(\partial^2 \varphi(\beta t_a + U(\tau - t_a))/\partial Y^2) > 0$ and the distributor is the leader in the supply chain. If $(\partial^2 I/\partial c_1^{+2}) > 0$, $(\partial\varphi(\beta\tau)/\partial Y) < 0, (\partial\varphi(U\tau)/\partial Y) > (c^- - (c_1^+ + c^-)F(U))\tau$, and $m \le \rho \le s$, then $u(t) = \beta$ for $0 \le t < t_a$, $u(t) = U$ for $t_a \le t \le \tau$, and $c_1^+ = \rho$ constitute a unique Stackelberg equilibrium in the differential inventory outsourcing game for $0 \le t \le \tau$. ■

Theorems 2 and 3 demonstrate that the distributor's charges are generally higher than the marginal cost m, as a result of competition. However, these charges decrease and thus implied double marginalization reduces compared to those from Theorem 1 for any unit of inventory sent to the distributor when there is only one period to go. This implies that a higher level of uncertainty when there are two periods to go makes the distributor more sensitive to the manufacturer's cost dependent production policies.

Finally, the cases defined in Lemma 6 which are described by boundary controls and no switching points are immediate: the equilibrium price should be at the maximum value.

C. Time Consistency Considerations

The equilibrium in Stackelberg open-loop games is defined as strongly time consistent if at a time point t its truncated part is an equilibrium for the subgame, independent of the conditions regarding state variables at t [5]. That is, an equilibrium can be time inconsistent if during the game when the state variables change the leader may modify the plan chosen at the beginning to achieve optimality. Time inconsistency is an important consideration when dealing with either stochastic games characterized by continuously observed (updated) and controllable states or deterministic differential games as the states are always observable. In our two-period game model, the leader can choose or modify the control only at the beginning of each period but not inside the two periods. Therefore, the open-loop equilibrium in our model is intrinsically strongly time consistent. Specifically, in the first period, the parties are not able to observe the system state and reevaluate their optimal responses. There is a state update at the beginning of the second period, but our game becomes a deterministic differential game in that period. An important property ensuring the time consistency of open-loop Stackelberg equilibria for such a game, referred to as uncontrollable game [22], is that the leader cannot manipulate the equilibrium through its control variables. This property is explicitly observed in Theorem 1, where a change of the control variable of one party does not affect the decision of the other party. Of course, if we modify the assumptions made in this paper, for example, if the leader were able to change the inventory charge during the second period, then the equilibrium would be only weekly time consistent. As shown in Theorem 1, the first-order optimality condition of the distributor depends on the inventory level (state variable), which implies that an open-loop Stackelberg equilibrium would be weakly time consistent at the time interval if the inventory charge could be changed during this interval [5].

V. Example and Computational Results

Following Fisher *et al.* [7], [8] examples, we consider a case when the demand is the same during the entire production horizon, i.e., its realization $D_2 = D_1$ is accurately known by time $t = \tau$, if τ is chosen so that $(\tau/\tau + T) \ge 0.2$ and thereby at least 20% of the total demand is observed at $t = \tau$. Specifically, consider a uniform distribution, $f_1(D) = 1/A$, $A < U$, and $f_2(D_2|D_1) = \delta(D_2 - D_1)$, where $\delta(\cdot)$ is the Dirac delta function. Then, objective function (10) takes the following form, as shown in the equation at the bottom of the page, and shown in the first equation at the bottom of the next page. Next, if $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=0} < c^-\tau$ and $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=\beta\tau} > 0$, then $u(t) = \beta$ for $0 \le t < t_b$ and $u(t) = 0$ for $t_b \le t \le \tau$ (see Lemma 6 and Theorem 3), therefore, with the aid of the last expression, we have equation (24) at the bottom of the next page, where switching point t_b is a function of c_1^+ as defined by (22).

Under the same uniform distribution, (22) takes the following form

$$\int_{t_b}^{\tau} \left[(c_1^+ + c^-)\frac{\beta t_b}{At} - c^- \right] dt = -\frac{\partial\varphi(Y(\tau))}{\partial Y(\tau)}\,|_{Y(\tau)=\beta t_b} \tag{25}$$

where $\varphi(Y(\tau))$ is determined by (11). Applying uniform distribution to (11), we find the third equation at the bottom of the next page.

$$\begin{aligned} I &= \int_0^{\tau}\int_0^{(Y(t)/t)} (m - c_1^+)\,(Y(t) - D_1 t)\, f_1(D_1) dD_1 dt + \eta(Y(\tau)) \\ &= \int_0^{\tau}\int_0^{(Y(t)/t)} (m - c_1^+)\,(Y(t) - D_1 t)\, \frac{1}{A} dD_1 dt + \eta(Y(\tau)) \\ &= \int_0^{\tau} (m - c_1^+)\frac{Y^2(t)}{2tA} dt + \eta(Y(\tau)) \end{aligned}$$

Thus, (25) takes the following form shown in equation (26) at the bottom of the page. Though (26) is transcendental, that is it cannot be resolved explicitly in t_b, one can derive an expression for $(dt_b/dc_1^+) = \Gamma(c_1^+)$ by implicitly differentiating (26) and assuming that t_b depends on c_1^+. In addition, (26) can be solved explicitly in c_1^+. Consequently, given (dt_b/dc_1^+), equilibrium cost $c_1^+ = \gamma$ is found by solving (23). Specifically, differentiating objective function I (defined by (24)) and assuming again that t_b depends on c_1^+, we find

$$\begin{aligned}\frac{\partial I}{\partial c_1^+} = &-\frac{1}{4}\frac{1}{A}\beta t_b\\ &\times\left(-2t_b m\frac{d\beta}{dc_1^+} - 4t_b m\ln\tau\frac{d\beta}{dc_1^+}\right.\\ &\quad - 4\beta m\ln\tau\frac{dt_b}{dc_1^+} + 4t_b m\ln t_b\frac{d\beta}{dc_1^+}\\ &\quad + 4\beta m\ln t_b\frac{dt_b}{dc_1^+} + 2t_b c_1^+\frac{d\beta}{dc_1^+} + \beta t_b\\ &\quad + 4t_b c_1^+\ln\tau\frac{d\beta}{dc_1^+} + 4\beta c_1^+\ln\tau\frac{dt_b}{dc_1^+} + 2\beta t_b\ln\tau\\ &\quad - 4t_b c_1^+\ln t_b\frac{d\beta}{dc_1^+} - 4\beta c_1^+\ln t_b\frac{dt_b}{dc_1^+}\\ &\quad\left. - 2\beta t_b\ln t_b\right),\end{aligned} \tag{27}$$

where $(d\beta/dc_1^+) = -(c^-/(c_1^+ + c^-)^2)/f_1(\beta)$ is obtained from Lemma 1, and c_1^+ and (dt_b/dc_1^+) can be substituted from (26). If the contract cannot be changed during the production horizon, then c_1^+ is set equal to c_2^+ when solving (27). Otherwise, if a change is possible at the end of the first period, then c_2^+ should be set at maximum, i.e., $c_2^+ = s$ when solving (27) in c_1^+. This implies that the equilibrium price depends not only on leadership in the supply chain, but also on the type of contract. We note that (26) and (27) constitute a system of two algebraic equations in two unknowns, equilibrium inventory charge c_1^+ and switching point t_b. We next verify numerically that $m \le c_1^+ \le s$, $(\partial\varphi(0)/\partial Y) < c^-\tau$ and $(\partial\varphi(\beta\tau)/\partial Y) > 0$, as required by Theorem 2.

1) Computational Results: We conduct two numerical studies, which show the effect of the unit shortage cost and that of the period duration, respectively, on the switching point and equilibrium inventory charge. For the first study, we let c^- run

$$\begin{aligned}\eta(Y(\tau)) = &\int_0^{(Y(\tau)/T)} f_1(D_1)(m - c_2^+)\left((Y(\tau) - D_1\tau)(T-\tau) - \frac{D_1}{2}(T-\tau)^2\right)dD_1\\ &+ \int_{(Y(\tau)/T)}^{(Y(\tau)/\tau)} f_1(D_1)(m - c_2^+)\frac{(Y(\tau) - D_1\tau)^2}{2D_1}dD_1\\ = &\frac{1}{A}(m - c_2^+)\frac{Y^2(\tau)}{4}\left[\frac{(3T-\tau)(T-\tau)}{T^2} + 2\ln\frac{T}{\tau} - 4\left(1 - \frac{\tau}{T}\right) + \left(1 - \frac{\tau^2}{T^2}\right)\right].\end{aligned}$$

$$\begin{aligned}I = &\int_0^{\tau}(m - c_1^+)\frac{Y^2(t)}{2tA}dt + \eta(Y(\tau))\\ = &\int_0^{t_b}(m - c_1^+)\frac{\beta^2 t}{2A}dt + \int_{t_b}^{\tau}(m - c_1^+)\frac{\beta^2 t_b^2}{2tA}dt + \eta(Y(\tau))\\ = &(m - c_1^+)\frac{\beta^2 t_b^2}{2A}\left(\frac{1}{2} + \ln\tau - \ln t_b\right)\\ &+ \frac{1}{A}(m - c_2^+)\frac{\beta^2 t_b^2}{4}\left[\frac{(3T-\tau)(T-\tau)}{T^2} + 2\ln\frac{T}{\tau} - 4\left(1 - \frac{\tau}{T}\right) + \left(1 - \frac{\tau^2}{T^2}\right)\right],\end{aligned} \tag{24}$$

$$\begin{aligned}-\frac{\partial\varphi(Y(\tau))}{\partial Y(\tau)}\Big|_{Y(\tau)=\beta t_b} &= \psi(T)\\ &= \frac{1}{A}\left(c_2\beta t_b\ln\left(\frac{\tau}{T}\right) + c^-\beta t_b\ln\left(\frac{\tau}{T}\right) + c^-U\tau\ln\left(\frac{T}{\tau}\right)\right)\end{aligned}$$

$$(c_1^+ + c^-)\frac{\beta t_b}{A}(\ln\tau - \ln t_b) - c^-(\tau - t_b) = \frac{1}{A}\left(c_2\beta t_b\ln\left(\frac{\tau}{T}\right) + c^-\beta t_b\ln\left(\frac{\tau}{T}\right) + c^-U\tau\ln\left(\frac{T}{\tau}\right)\right). \tag{26}$$

TABLE I
COMPUTATIONAL RESULTS FOR THE FIRST STUDY

c^-	c_1^+	t_b	$\frac{\partial\varphi(Y(\tau))}{\partial Y(\tau)}\Big\vert_{Y(\tau)=\beta t_b}$	$\frac{\partial\varphi(Y(\tau))}{\partial Y(\tau)}\Big\vert_{Y(\tau)=0} - c^-\tau$
1	3.347	4.392	49.185	-27.997
1.5	3.71	4.973	59.823	-41.995
2	4.114	5.535	65.839	-55.993
2.5	4.555	6.08	68.83	-69.991
3	5.03	6.608	69.767	-83.99
3.5	5.539	7.123	69.255	-97.988
4	6.078	7.626	67.692	-111.986
4.5	6.646	8.112	65.344	-125.984
5	7.244	8.603	62.396	-139.983
5.5	7.869	9.08	58.98	-153.981
6	8.522	9.552	55.193	-167.979
6.5	9.202	10.02	51.103	-181.977
7	9.908	10.486	46.767	-195.976
7.5	10.641	10.952	42.224	-209.974
8	11.401	11.419	37.507	-223.972
8.5	12.187	11.888	32.645	-237.97
9	12.998	12.362	27.659	-251.969
9.5	13.834	12.841	22.572	-265.967
10	14.693	13.327	17.407	-279.965

over a range of values among which the solution exists, and set the other parameters as follows:

$$\{U = 100, T = 30, \tau = 15, A = 80, m = 3, s = 25, c_2 = 25\}.$$

The corresponding values of the equilibrium c_1^+ and switching point t_b are described in the following table. Since the optimal production rate depends on $(\partial\varphi(Y(\tau))/\partial Y(\tau))$, which indicates whether the switching point exists: if $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=0} < c^-\tau$, but $(\partial\varphi(Y(\tau))/\partial Y(\tau))\,|_{Y(\tau)=\beta\tau} > 0$, then $u(t) = \beta$ for $0 \le t < t_b$ and $u(t) = 0$ for $t_b \le t \le \tau$ (otherwise, $u(t) = 0$ for $0 \le t \le \tau$, see Lemma 6), we also present these characteristic values in Table I to show that the optimality conditions hold throughout our experiments.

From Table I, we observe that all the conditions of Theorem 2 are satisfied. Furthermore, as one would expect, the greater the unit shortage cost, the later the switching point t_b and thereby the longer the production duration. This increases the inventory surplus, which the leader makes use of monopolistically by increasing the equilibrium cost he charges, as illustrated in Figs. 3 and 4.

For the second study, we let the unit shortage cost c^- be constant and change the review period τ. The other parameters are set as follows:

$$\{U = 100, T = 30, c^- = 6, A = 80, m = 3, M = 25, c_2 = 25\}.$$

The results are presented in Table II.

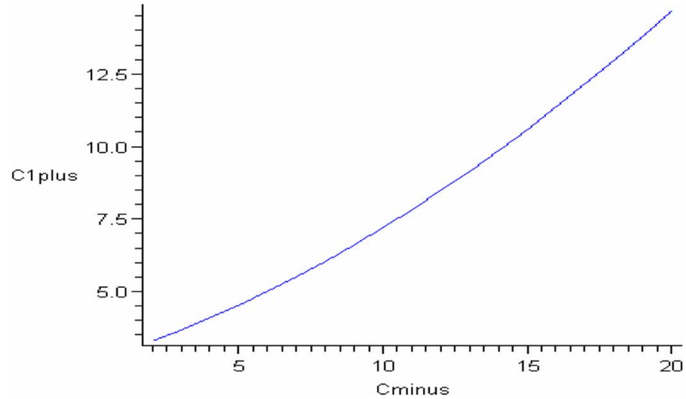

Fig. 3. The inventory charge as a function of the unit backlog cost.

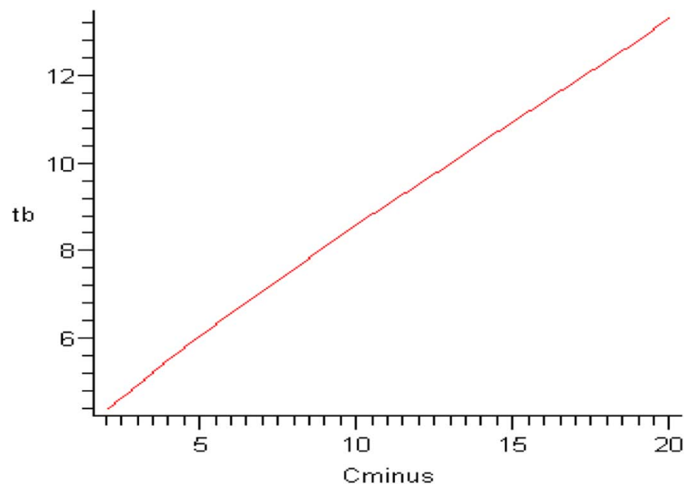

Fig. 4. The switching point as a function of the unit backlog cost.

TABLE II
COMPUTATIONAL RESULTS FOR THE SECOND STUDY

τ	c_1^+	t_b	$\frac{\partial\varphi(Y(\tau))}{\partial Y(\tau)}\Big\vert_{Y(\tau)=\beta t_b}$	$\frac{\partial\varphi(Y(\tau))}{\partial Y(\tau)}\Big\vert_{Y(\tau)=0} - c^-\tau$
10	7.796	5.952	56.893	-139.983
11	7.94	6.636	58.289	-153.981
12	8.084	7.337	59.332	-167.979
13	8.228	8.056	60.063	-181.977
14	8.374	8.794	60.513	-195.978
15	8.522	9.552	60.706	-209.974
16	8.672	10.331	60.665	-223.973
17	8.826	11.135	60.405	-237.97
18	8.984	11.963	59.939	-251.969
19	9.147	12.82	59.278	-265.967
20	9.316	13.709	58.426	-279.965

From Table II, we observe that all the conditions of Theorem 2 are satisfied again. Furthermore, the intuition behind the effect of the review period length is similar to that of the shortage cost. Specifically, the longer the period, the later the switching point t_b (the longer the production duration) and the higher the

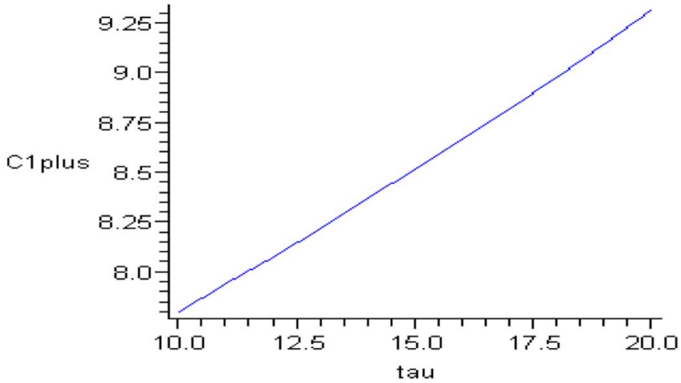

Fig. 5. The inventory charge as a function of the review period length.

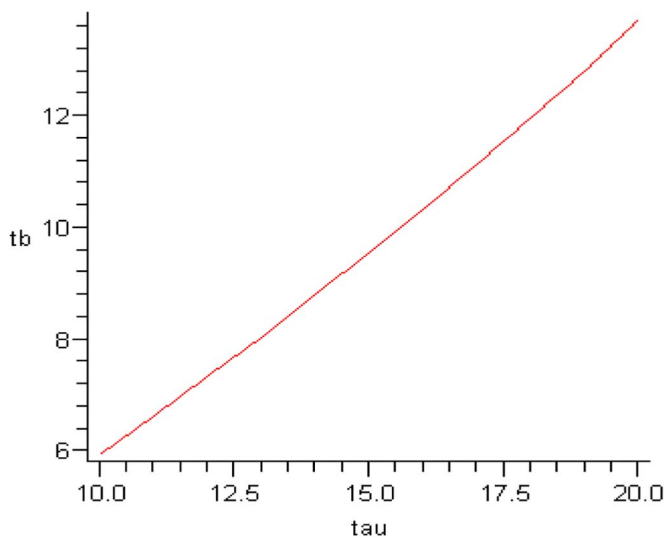

Fig. 6. The switching point as a function of the review period length.

equilibrium cost charged by the leader for the inventories stored at his warehouse, as illustrated in Figs. 5 and 6.

Overall, we observe that prolonged review period and thereby higher uncertainty lead to later switching points, longer production periods, and lager inventories. Although increasing stocks under higher uncertainty is known in static supply chains (see, for example, [18]), we determine when and how the production rate should be changed as a function of demand distributions as a result of intrasupply-chain competition. Such a dynamic effect of stochastic demands on production is especially important as many modern production systems allow for production to be adjusted at any point of time rather than just once at the very beginning (as in static models) or at each point of time but with no updates (as in deterministic differential models).

VI. Conclusion

In this paper, we provide analytical results for a two echelon supply chain involving a manufacturer and a distributor utilizing inventory outsourcing with an uncertain demand. We assume that the probability distribution of the demand rate at the distributor's site is known, but the inventory level can only be observed by the manufacturer intermittently. The optimal production control that minimizes a linear combination of the expected surplus and shortage costs over the planning horizon is shown to be piecewise constant. In addition, the optimal production levels and control switching points can be determined as functions of the demand rate distribution. We demonstrate that even for the seemingly simple one-part-type system with a single inventory review, the inclusion of the random demand still leads to a nontrivial optimal production control for the manufacturer. As to the distributor, his charge for handling manufacturer's inventories is determined by his leadership in the supply chain and depends on the remaining number of periods to go. While the desire to charge as much as possible for handling inventories at the last period prevails (it corresponds to the severe double marginalization), the distributor reduces the charge (a situation corresponding to the implied double marginalization) when there are two periods to go. This, in turn, affects the optimal production policy chosen by the manufacturer. This dynamic effect is due to the uncertainty at the first production period, which is different from the second, where the demand is no longer random. Our numerical computations show that increasing the length of the first inventory review period, thus higher uncertainty, results in both larger inventories and higher charges for handling them.

In summary, this paper shows that under intrasupply-chain competition and stochastic demands the dynamic production policies depend on demand distributions and the costs that the distributor charges. These policies are based on precise timing, which demonstrates the importance of developing supply chain models employing a continuous-time framework, i.e., a differential game approach. This paper is only a preliminary step in analyzing supply chain dynamics under uncertainty. Many issues, such as the price and stock dependency of demands, different types of leadership in the differential game, multiperiod, and infinite horizon games should be important directions for future research.

Appendix

Proof of Lemma 1: Differentiating the condition $\psi(t)=0$ over r and taking into account (17), we find $F_1\left(Y(t)/t\right) = (c^-/c_1^+ + c^-)$.

Thus, $\beta = (Y(t)/t)$ and, therefore, $Y(t)=\beta t$ and $\dot{Y}(t) = u(t) = \beta$ for $t \in r$. ■

Proof of Lemma 2: Consider the following solution for the state variables $u(t)=0$, $Y(t)=X(0)$ for $0 \le t < (X(0)/\beta)$; $u(t)=\beta$, $Y(t)=\beta t$ for $(X(0)/\beta) \le t < t_a$; $u(t)=U$, $Y(t)=\beta t_a + U(t-t_a)$ for $t_a \le t \le \tau$; and for the co-state variables $\psi(t) = -\int_t^{(X(0)/\beta)} \left[(c_1^+ + c^-)F_1\left(X(0)/t\right) - c^-\right] dt$ for $0 \le t < (X(0)/\beta)$; $\psi(t)=0$ for $(X(0)/\beta) \le t < t_a$; and $\psi(t) = \int_{t_a}^{t} [(c_1^+ + c^-)F_1\left(\beta t_a + U(t-t_a)/t\right) - c^-] dt$ for $t_a \le t \le \tau$.

If this solution is feasible and satisfies optimality conditions (19), then it is optimal. The feasibility, $t_a > (X(0)/\beta)$ and $(\partial\varphi(Y(\tau))/\partial Y(\tau))\big|_{Y(\tau)=\beta t_a + U(\tau - t_a)} < 0$, is imposed in the statement of this lemma. The optimality conditions are verified as follows. It is easy to observe that because $X(0) < \beta\tau$ and $F_1(\beta) = (c^-/c_1^+ + c^-)$, we have $(c_1^+ + c^-)F_1\left(X(0)/t\right) - c^- > 0$ for $0 \le t < (X(0)/\beta)$ and thus $\psi(t) = -\int_t^{(X(0)/\beta)} \left[(c_1^+ + c^-)F_1\left(X(0)/t\right) - c^-\right] dt < 0$. That is, the second optimality condition $u(t)=0$ in (19) holds.

Similarly, $\psi(t) = \int_{t_a}^{t} \left[(c_1^+ + c^-)F_1(\beta t_a + U(t-t_a)/t) - c^-\right] dt$ for $t_a < t \leq \tau$, and thus the first optimality condition $u(t) = U$ in (19) holds. The third condition in (19) is explicit: $\psi(t) = 0$ for $(X(0)/\beta) \leq t < t_a$ and thus $u(t) = \beta$. ■

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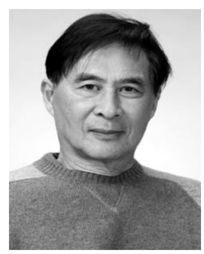

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