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A supply chain under limited-time promotion: The effect of customer sensitivity

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Abstract

We consider a two-echelon supply chain with a supplier and a retailer facing stochastic customer demands. The supplier is a leader who determines a wholesale price. In response, the retailer orders products and sets a price which affects customer demands. The goal of both players is to maximize their profits. We find the Stackelberg equilibrium and show that it is unique, not only when the supply chain is in a steady-state but also when it is in a transient state induced by a supplier's promotion. There is a maximum length to the promotion, however, beyond which the equilibrium ceases to exist. Moreover, if customer sensitivity increases, then the wholesale equilibrium price decreases, product orders increase and product prices drop. This effect, well-observed in real life, does not, however, necessarily imply that the promotion is always beneficial. Conditions for the profitability of a limited-time promotion are shown and analyzed numerically. We discuss both open-loop and feedback policies and derive the conditions necessary for them to remain optimal under stochastic demand fluctuations.

Keywords: Supply chain management; Gaming; The maximum principle

1. Introduction

Surveys published in Progressive Grocer steadily report that manufacturing, wholesale, and chain store executives claim that promotional programs are a top concern for their firms. Though large manufacturers traditionally dominate trade deals, retailers armed with abundant information on profitability, product movement, and customer demand for a class of goods are developing sophisticated purchase and storage policies to take advantage of the trade promotions available from manufacturers. A retailer, for instance, may engage in "forward buying", that is, purchasing more goods during a promotional period than he expects to sell (Zerrillo and Iacobucci, 1995). On the other hand, increased use of promotions (e.g., weekend and holiday tariffs)

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enhances customer price sensitivity or leads to more price anticipation (Jedidi et al., 1999; Kopalle et al., 1999). This paper analyzes such phenomena and provides formal rationales for complex purchase and inventory policies under increased customer sensitivity.

We address the continuous-time behavior of a two-echelon supply chain facing limited-time promotion and stochastic demand. There is a leader – a supplier or wholesaler with ample capacity – and a follower – the retailer. When the supplier sets a wholesale price, the retailer commits to purchase a certain quantity. Both desire to maximize their profits. The contract between these players is of the rolling-horizon type which implies that purchase orders can be periodically updated within certain limitations (Anupindi and Bassok, 1998). If demand as well as the chain parameters are steady, then there is a static Stackelberg solution to this two-player game. However, if the demand changes, the Stackelberg strategy becomes dynamic. It is important to note that the Stackelberg strategy is applied when there is power/information asymmetry in the supply chain. This strategy is especially reasonable when one of the parties knows only his own cost function but the other party knows both cost functions.

We study the effect of changes in customer price sensitivity on a supply chain operating under a limited-time promotion. The promotion may be initiated by the leader, confirmed or prompted by the follower, or it may be imposed by special business conditions. A typical initiative is an advertisement about a limited-time promotion included in a routine advertising campaign. Such an ad, offering special prices, as opposed to a gradual price discount, may substantially affect customer demand if the commodity under sale is a relatively new or improved modification of a well-known product. An example of special business conditions is a national holiday, Christmas being among the most prominent. Empirical studies show that consumers are more price sensitive during periods of high demand such as Christmas, Thanksgiving and Weekends (see, for example, Chevalier et al., 2003; Bils, 1989; Warner and Barsky, 1995). In the UK, for example, Christmas sales of consumer electronics may reach up to 40% of the annual sales. Such a shock in demand stresses the supply chain, due to an instantaneous change in customer price sensitivity during holidays. This change, which can cause customers to buy more than they usually would, indeed, more than they would normally buy even during a regular promotion, increases the demand potential, $a(t)$ as well. To illustrate this phenomenon, one can view demand $d(t)$ for a product as a function of the current product price $p(t)$, the list price P and the customer price sensitivity $b(t)$, $d(t) = g(t) + b(t)(P - p(t))$, where $g(t)$ is the demand under anticipated list pricing, $p(t) = P$. Then, by denoting the demand potential, $a(t) = g(t) + b(t)P$, we observe that this function is equivalent to the classical linear demand function, $d(t) = a(t) - b(t)p(t)$. This is to say, if customer sensitivity $b(t)$ increases during a limited-time promotion, the demand potential $a(t) = g(t) + b(t)P$ may increase as well even if $g(t)$ remains unchanged. This also implies that sales during a period of increased customer sensitivity and, as a result, increased demand elasticity may become more efficient than those offered during regular times. For example, if customer price sensitivity $b(t)$ increases during a limited-time period by K units per dollar and $p(t) \leqslant P$, then the positive increment in demand, $b(t)(P - p(t))$, includes K additional product units for each dollar discounted in price $p(t)$ compared to sales offered at other times.

The retailer's response to temporary vendor trade promotions, including offering a retail pricing promotion to customers, has stimulated considerable research (see, for example, Silver et al., 1998; reviews by Arcelus and Srinivasan, 1995; Tersine and Barman, 1995). Kopalle et al. (1999) are among the first in marketing who suggested a normative model which includes the effect of promotions on purchasing strategies in a discrete-time dynamic Stackelberg game. However, they model the discounting rather than changes in regular prices and use empirical price reaction functions for numerical optimization. The model accounts for competition between brands but allows no retailer forward buying which, as they acknowledge, would be an important extension.

The retailer's ability to collect detailed information about customer purchasing behavior and the ease of changing prices due to new technologies (including Internet and IT) has engendered extensive research into dynamic pricing in general and continuous-time pricing strategies in particular. Increasing attention has been paid to dynamic pricing in the presence of inventory considerations (see, for example, a recent survey by Elmaghraby and Keskinocak, 2003) and to coordinated pricing and production/procurement decisions (see surveys by Chan et al., 2003; Yano and Gilbert, 2002; Cachon, 2003). However, despite this range of research interests, relatively few studies are devoted to the continuous interaction between dynamic retail prices, inventory-related costs and wholesale prices in supply chains, i.e., to a dynamic continuous-time game between supply chain members.

Due to mathematical difficulties inherent in differential games, i.e., games involving decisions that have to be made continuously, the supply chain management literature in this area has been primarily concerned only with the application of deterministic differential models (Cachon and Netessine, 2004). Jorgenson (1986) derives an open-loop Nash equilibrium under a channel setting and static deterministic demand with demand potential $a(t)$ and customer sensitivity $b(t)$ being constant. Eliashberg and Steinberg (1987) use the open-loop Stackelberg solution concept in a game with a manufacturer and a distributor (both of unlimited capacity) involving quadratic seasonal demand potential $a(t)$ and constant sensitivity $b(t)$. Assuming that the wholesale price the manufacturer charges the distributor is constant (no promotions) and that no backlogs are allowed, they investigate the impact of the quadratic seasonal pattern upon the various policies of the channel. They acknowledge that demand uncertainty jointly with stockout costs may change the results and suggest supplementing the proposed procedure with a sensitivity analysis of the solution found. Desai (1992) allows for demand potential to change with an additional decision variable. To address seasonal demands, he later suggests a numerical analysis for a general case of the open-loop Stackelberg equilibrium under a sine form of $a(t)$, constant customer sensitivity $b(t)$ and unlimited manufacturer and retailer capacities (Desai, 1996).

In contrast to the above papers, which study open-loop Stackelberg strategies for uncapacitated supply chains characterized by deterministic gradual change in demand potential and constant customer price sensitivity, we consider:

- instantaneous exogenous change in the customer price sensitivity, $b(t)$, which shocks the demand potential, $a(t)$, as well; the wholesale price is endogenous and changes because of the change in price sensitivity;
- open-loop and feedback Stackelberg strategies for inventory levels observable during the planning horizon;
- a sensitivity analysis to determine the effect of stochastic demand fluctuations;
- a retailer characterized by a limited processing capacity.

After formulating the model in Section 2, we study the problem under steady-state conditions of no promotions in Section 3. In this section we first derive an open-loop equilibrium for deterministic demand, then suggest its feedback form and discuss the conditions for this feedback policy to remain optimal despite stochastic changes in demand. Section 4 is devoted to the supply channel behavior under a limited-time promotion. We show that the promotion transforms the channel from a steady-state to a transient state. In Section 4 we begin, as in Section 3, with a deterministic open-loop solution to gain insight into the optimal behavior of the channel. Then an optimal feedback policy and the effect of random demand are formalized. We show that purchasing more goods during a promotional period than the retailer expects to sell (forward buying) is optimal for the retailer along with cutting retail prices. This requires simultaneous control of marketing and production policies. The fine-tuning of the retailer strategy is to build some backlog demand around the starting point of the promotion and some inventory around the ending point of the promotion. Furthermore, we find that if the customer sensitivity, $b(t)$, increases during a promotion, the equilibrium wholesale price decreases, product orders and demand increase and product prices drop. This effect, well-observed in real life (Chevalier et al., 2003; Warner and Barsky, 1995), does not, however, necessarily imply that the promotion is beneficial for the players. A necessary and sufficient condition for a limited-time promotion to be profitable is shown and analyzed numerically. Section 5 summarizes our results.

2. The model

Consider a two-echelon supply chain with a supplier who distributes a single product-type to a retailer at a wholesale price. The supplier chooses a wholesale price. In response, the retailer orders products and sets up a product price that affects customer demand. The retailer only observes current demand; future demands are unknown. The supplier and the retailer incur linear processing costs for ordering, transporting and handling the product at a certain rate referred to as the processing rate. Since the supplier replenishes the products by ordering from an outside source with ample supply and processing capacity, he can supply all the retailer's orders without incurring any stockout. On the other hand, the retailer's processing rate is bounded and excess demand is backlogged, which is why inventory/backlog costs could develop.

2.1. Notations

$X(t)$ retailer inventory level at time t, a state variable
$u(t)$ order quantity processed by the retailer at time t (processing rate), a decision variable
U retailer maximum processing rate
$d(t) = a(t) - b(t)p(t) + e_t$ customer demand rate for the retailer, where $a(t)$ is the demand potential at time t, $b(t)$ is the price effect on the customer demand (customer sensitivity) at t, $e_t = e_k$, for $t^k \leqslant t < t^{k+1}$, $k = 0, 1, 2, \ldots$, $t^0 = 0$ and e_k is a random disturbance observable at $t = t_k$, and characterized by a bounded distribution with $E[e_k] = 0$
$p(t)$ retail price at time t, a decision variable
$w(t)$ unit wholesales price charged by the supplier, a decision variable
h^+, h^- product unit holding and backlog costs respectively incurred per time unit by the retailer
c_r, c_s product unit processing costs incurred by the retailer and supplier respectively

2.2. Statement of the problem

Let a typical rolling-horizon contract formalize the supplier and the retailer relationship. This implies an infinite planning horizon and a period T that characterizes the contract. During T, mutual supplier–retailer commitments cannot be revised. Specifically, the supplier sets a constant wholesale price for a period, T. In response, the retailer commits to order fixed quantities with minor variations to cope with demand fluctuations within the period. If the demand is steady, this type of contract results in a steady-state that the commitments determine, i.e., constant wholesale price, as well as constant retailer order quantity, inventory level and product price. This steady-state can be deliberately disrupted if a limited-time promotional sale is initiated. With respect to this initiative, the supplier is expected to reduce the wholesale price from w_1 to w_2 for the promotional period of time $[t_s, t_f]$ to boost sales,

$$w(t) = \begin{cases} w_1, & t < t_s \quad \text{and} \quad t \geqslant t_f, \\ w_2, & t_s \leqslant t < t_f, \end{cases} \qquad w_1 \geqslant w_2. \tag{1}$$

In response to the change, the retailer also drops prices and increases order quantities. As a result, the chain is in a transient state for a period of time comprising the interval, $[t_s, t_f]$. Furthermore, since the promotion dates are either advertised or coincide with especially sensitive seasons (e.g. holidays), the price sensitivity of the customers, $b(t)$, during these dates increases:

$$b(t) = \begin{cases} b_1, & t < t_s \quad \text{and} \quad t \geqslant t_f, \\ b_2, & t_s \leqslant t < t_f, \end{cases} \qquad b_2 \geqslant b_1.$$

As noted in the introduction, this increase in price sensitivity increases the demand potential $a(t)$ during the promotion as well,

$$a(t) = \begin{cases} a_1, & t < t_s \quad \text{and} \quad t \geqslant t_f, \\ a_2, & t_s \leqslant t < t_f. \end{cases}$$

That is, if $b_2 > b_1$, then from $a_1 = g + b_1 P$ and $a_2 = g + b_2 P$, we have $a_2 > a_1$. Since the effect of the customer sensitivity on demand potential is not necessarily linear, we relax the linearity and employ a more general assumption with respect to the demand potential

$$\frac{a_1}{b_1} > \frac{a_2}{b_2},$$

which ensures that the demand elasticity, $-\frac{\partial d(t)}{\partial p(t)} \frac{p(t)}{d(t)} = \frac{p(t)}{\frac{a(t)}{b(t)} - p(t)}$, and thus the efficiency of price cuts, increases. Note, that this assumption is always met for any linear function $a(t) = g + b(t)P$, if $b_2 > b_1$. Therefore we will employ this special case in the paper as well whenever it is possible to gain more insights into the problem.

The effect of an increase in customer sensitivity occurs only if the promotional time interval, $[t_s, t_f]$, is much shorter than the regular contract period T, which is typically the case with limited-time promotions as well as national holidays. Therefore we consider a period of time $[0, T]$ such that the supply chain, which was in steady-state at the beginning of the period, will have enough time after the promotional interval to return to this state by time T.

2.2.1. Supplier's problem

We assume that the supplier has ample capacity and that his dynamics are straightforward: produce (supply) exactly according to retailer orders $u(t)$ to maximize expected profits by choosing regular, w_1, and promotional, w_2, wholesale prices:

$$J_s = E\left[\int_0^T [w(t)u(t) - c_s u(t)]\,dt\right] \to \max \tag{2}$$

$$\text{s.t.} \quad w(t) \geqslant 0, \tag{3}$$

where $w(t)$ is determined by (1); the first term in the objective function (2) presents wholesale revenues over time; and the other term presents supplier processing costs over time.

2.2.2. Retailer's problem

The retailer also wants to maximize expected profits by selecting proper order quantities and product prices $\{u(t), p(t), 0 \leqslant t \leqslant T\}$:

$$J_r = E\left[\int_0^T [p(t)(a(t) - b(t)p(t) + e_t) - c_r u(t) - w(t)u(t) - h(X(t))]\,dt\right] \to \max \tag{4}$$

$$\text{s.t.} \quad \dot{X}(t) = u(t) - (a(t) - b(t)p(t) + e_t); \tag{5}$$

$$0 \leqslant u(t) \leqslant U; \tag{6}$$

$$d(t) = a(t) - b(t)p(t) + e_t \geqslant 0; \tag{7}$$

$$p(t) \geqslant 0, \tag{8}$$

where the first term in the objective function (4) presents revenues of the retailer from the sales $d(t) = a(t) - b(t)p(t) + e_t$; the second term reflects retailer processing costs; and the third is the cost of purchasing from the supplier at the wholesale price. The last term in (4) accounts for inventory costs where

$$h(X(t)) = h^+X^+(t) + h^-X^-(t),$$
$$X^+(t) = \max\{X(t), 0\} \text{ and } X^-(t) = \max\{-X(t), 0\},$$

which are due to the bounded capacity (6) of the retailer. With respect to the inventory balance equation (5), if the cumulative processing rate at time t is greater than the cumulative demand at t, then the inventory holding cost is incurred at $t, h^+X(t)$, otherwise the backlog cost $h^-X^-(t)$ is incurred.

In this work we use the Stackelberg approach to solve the supplier and retailer problems with the supplier acting as the leader and the retailer acting as the follower. The next section determines the Stackelberg equilibrium when the supply chain is in a steady-state.

3. The steady-state analysis

In this section we derive a Stackelberg equilibrium under steady, rolling-horizon, contract conditions characterized by constant wholesale prices, retailer orders and inventory levels which are naturally kept at zero level. This implies that we consider a subperiod during which no promotion initiative is expected.

3.1. Deterministic demand

We start off by considering the case when the demand is deterministic, i.e., $e_t = 0$ for any t. Then to derive deterministic equilibrium between the two players, the supplier and the retailer, we use the maximum-principle-based optimality conditions (Simaan and Cruz, 1973). Specifically, given a wholesale price, $w(t)$, we first derive the optimal retailer's response function by maximizing the Hamiltonian

$$H(t) = p(t)(a(t) - b(t)p(t)) - c_r u(t) - w(t)u(t) - h(X(t)) + \psi(t)(u(t) - a(t) + b(t)p(t)), \tag{9}$$

with respect to the price $p(t)$ and processing rate $u(t)$, where the co-state variable $\psi(t)$ is determined by the co-state differential equation

$$\dot{\psi}(t) = \begin{cases} h^+, & \text{if } X(t) > 0; \\ h^-, & \text{if } X(t) < 0; \\ h \in [-h^-, h^+], & \text{if } X(t) = 0. \end{cases} \tag{10}$$

If the supply chain system is at the same steady-state at $t = 0$ and $t = T$, i.e., it is characterized by steady demand potential $a(0) = a(T)$, customer sensitivity $b(0) = b(T)$, wholesale price $w(0) = w(T)$, and retailer's inventory state $X(0) = X(T)$, then the co-state variable must be also the same at these points of time:

$$\psi(0) = \psi(T). \tag{11}$$

Maximizing the Hamiltonian with respect to $p(t)$, i.e., considering

$$H_p = p(t)(a(t) - b(t)p(t)) + \psi(t)b(t)p(t),$$

subject to (7) and (8) we readily find

$$p(t) = \begin{cases} \frac{a(t)}{b(t)}, & \text{if } a(t) + b(t)\psi(t) > 2a(t); \\ \frac{a(t)+b(t)\psi(t)}{2b(t)}, & \text{if } 0 \leqslant a(t) + b(t)\psi(t) \leqslant 2a(t); \\ 0, & \text{if } a(t) + b(t)\psi(t) < 0. \end{cases} \tag{12}$$

Similarly, by maximizing the $u(t)$-dependent part of the Hamiltonian, $H_u = (-c_r - w(t) + \psi(t))u(t)$, subject to (6), we find

$$u(t) = \begin{cases} U, & \text{if } \psi(t) > c_r + w(t); \\ 0, & \text{if } \psi(t) < c_r + w(t); \\ a(t) - b(t)p(t), & \text{if } \psi(t) = c_r + w(t). \end{cases} \tag{13}$$

Note, that the third condition in (13), which presents the case of an intermediate processing rate, is obtained by differentiating the singular condition, $\psi(t) = c_r + w(t)$, along an interval of time where it holds. Then by taking into account (1) and (10), we conclude that this condition holds only if $X(t)=0$, i.e., $u(t) = d(t) = a(t) - b(t)p(t)$. Furthermore, this singular condition is feasible if in addition to the constraints (6)–(8), we have

$$d(t) = a(t) - b(t)p(t) \leqslant U. \tag{14}$$

To derive the steady-state retailer's best response function, we assume steady sales. Specifically, we consider a subperiod of time $\tau \subseteq [0, T]$ characterized by no promotion, so that the customer sensitivity $b(t) = b$, wholesale price $w(t) = w$, and inventory $X(t) = X$ remain constant, for a period of time, $t \in \tau$, rather than identical only at $t = 0$ and $t = T$ as imposed by (11). As shown in the following lemma, this requirement implies that the dynamic system exhibits a static behavior characterized by constant retailer's pricing and processing rates as well as zero inventory levels.

Lemma 1. *If* $b(t) = b$, $a(t) = a$, $w(t) = w$, $X(t) = X$ *for* $t \in \tau$, $\tau \subseteq [0, T]$, *and* $0 \leqslant a - b(c_r + w) \leqslant 2U$, *then* $X = 0$, $t \in \tau$, *and the optimal retailer's processing and pricing policies are*:

$$u(t) = \frac{a - b(c_r + w)}{2} \quad \text{and} \quad p(t) = \frac{a + b(c_r + w)}{2b} \quad \text{for } t \in \tau \text{ respectively.}$$

Note that this solution is equivalent to that of a monopolist facing a linear demand and linear production costs.

Lemma 1 determines the optimal retailer's strategy in steady-state during a no-promotion period. To define the corresponding supplier's game in a steady-state over an interval of time, for example $[0,T]$, we substitute the best retailer's response from Lemma 1 for $\tau=[0,T]$ into the objective function (2):

$$E\left[\int_0^T [w(t)u(t)-c_{\mathrm{s}}u(t)]\,\mathrm{d}t\right]=\frac{a-b(c_{\mathrm{r}}+w)}{2}(w-c_{\mathrm{s}})T. \tag{15}$$

Note that the maximum of objective function (15) does not depend on the length of the considered interval T. Thus, we conclude with the following theorem for the supply chain, which is in a steady-state along an interval, $[0,T]$.

Theorem 1. *If $b(t)=b,\ a(t)=a,\ w(t)=w,\ X(t)=X$ for $t\in[0,T]$, and $0\leqslant a-b(c_{\mathrm{r}}+c_{\mathrm{s}})\leqslant 4U$, then the supplier's wholesale pricing policy $w^*(t)=\frac{a-b(c_{\mathrm{r}}-c_{\mathrm{s}})}{2b}$, and the retailer's processing $u^*(t)=\frac{a-b(c_{\mathrm{r}}+c_{\mathrm{s}})}{4}$ and pricing $p^*(t)=\frac{3a+b(c_{\mathrm{r}}+c_{\mathrm{s}})}{4b}$ policies constitute the unique Stackelberg equilibrium for $t\in[0,T]$.*

According to Lemma 1 and Theorem 1, the retailer's problem may have an optimal solution and the supply chain may be in a steady-state if the demand is non-negative at this state and the maximum processing rate is greater than the maximal demand

$$\frac{a}{b}\geqslant c_{\mathrm{r}}+c_{\mathrm{s}}\quad\text{and } a<U,$$

as assumed henceforth to simplify the presentation.

3.2. The effect of random demand disturbance

The sensitivity of the optimal solution determined in Theorem 1 to a bounded change in demand is intuitively clear. Indeed, according to the theorem, the best retailer's response is to maintain state and co-state variables at levels of $X(t)=0$ and $\psi(t)=c_{\mathrm{r}}+w$, respectively. Consequently, if random demand fluctuations are bounded so that the order quantities $u(t)$ and pricing policies $p(t)$ can be adjusted to keep $X(t)=0$, then the optimal policies found in Lemma 1 and Theorem 1 would still remain optimal. More precisely, the optimal open-loop equilibrium of Theorem 1 would transform in a feedback policy. To prove this fact formally, however, we need to redefine the deterministic optimality conditions (9)–(13) with respect to the random demand disturbances e_t. This is accomplished by considering a small variation of the optimal processing rate and price and declaring that no such variation can improve the objective function, i.e., $\delta J_{\mathrm{r}}\leqslant 0$.

Let $e_t(\xi)$ be a realization of the demand disturbance e_t. Denote the set of all possible realizations over the entire production horizon $\{\xi\}$ as R. Consequently, denote the set of all realizations ξ', $\xi'\in R$, which coincide with a realization, ξ, from the beginning of the production horizon through time t, as $R(t,\xi)$. That is,

$$R(t,\xi)=\{\xi'|\xi'\in R,\ \text{and } e_{\mathrm{s}}(\xi')=e_{\mathrm{s}}(\xi),\quad\text{for } 0\leqslant s\leqslant t\}.$$

Given a realization of demand disturbance, $e_t(\xi)$, and the corresponding inventory level $X(t,\xi)$ at time t, the stochastic conditions (that identify an optimal relationship between the processing rate $u(t,\xi)$, pricing $p(t,\xi)$ and co-state variable $\psi(t,\xi)$) for this realization are summarized as follows (see Appendix A for the derivation):

$$p(t,\xi)=\begin{cases}\dfrac{a(t)}{b(t)}, & \text{if } a(t)+e_t(\xi)+b(t)E_{R(t,\xi)}[\psi(t,\xi)]>2a(t);\\[2ex] \dfrac{a(t)+e_t(\xi)+b(t)E_{R(t,\xi)}[\psi(t,\xi)]}{2b(t)}, & \text{if } 0\leqslant a(t)+e_t(\xi)+b(t)E_{R(t,\xi)}[\psi(t,\xi)]\leqslant 2a(t);\\[2ex] 0, & \text{if } a(t)+e_t(\xi)+b(t)E_{R(t,\xi)}[\psi(t,\xi)]<0.\end{cases} \tag{16}$$

$$u(t,\xi)=\begin{cases}U, & \text{if } E_{R(t,\xi)}[\psi(t,\xi)]>c_{\mathrm{r}}+w(t);\\ 0, & \text{if } E_{R(t,\xi)}[\psi(t,\xi)]<c_{\mathrm{r}}+w(t);\\ a(t)+e_t(\xi)-b(t)p(t,\xi), & \text{if } E_{R(t,\xi)}[\psi(t,\xi)]=c_{\mathrm{r}}+w(t).\end{cases} \tag{17}$$

Comparing the deterministic pricing optimality conditions (12) with the stochastic conditions (16), one can readily observe that the disturbance $e_t(\xi)$ is naturally added to the demand potential in the stochastic case. The only actual difference is that the co-state variable in (12) is replaced with its expectation in (16). Similarly, one can verify the same difference between the optimal deterministic (13) and stochastic order processing conditions (17). It is significant that the last condition of (17) requires that $X(t,\xi)=0$ as was the case with (13). Moreover, it is this condition that induces a steady-state. Therefore, if there is a feasible policy which makes it possible to retain zero inventory level for the stochastic formulation, then the same type of equilibrium could still be achieved. Consequently, this policy can be viewed as a feedback policy for real-time orders and the best retailer's response will be very similar to that of Lemma 1 as shown in Lemma 2.

Lemma 2. *If $b(t)=b,\ a(t)=a,\ w(t)=w,\ X(t,\xi)=X(\xi)$ for $t\in\tau,\ \tau\subseteq[0,T],\ \xi\in R$, and the random disturbances $e_t(\xi)$ are such that $0\leqslant a+e_t(\xi)-b(c_r+w)\leqslant 2U$, then $X(t,\xi)=0$, for $t\in\tau$, and the optimal retailer's processing and pricing policies are*:

$$u(t,\xi)=\frac{a+e_t(\xi)-b(c_r+w)}{2}\quad \textit{and}\quad p(t,\xi)=\frac{a+e_t(\xi)+b(c_r+w)}{2b}\quad \textit{for } t\in\tau \textit{ respectively.}$$

Lemma 2 provides a condition when feedback inventory level can be kept at zero and thus demand disturbances do not accumulate over time. Using Lemma 2, we are now able to verify that though the retailer's processing and pricing rates change, the optimal supplier's strategy under bounded disturbances will remain the same as shown in Theorem 2.

Theorem 2. *If $b(t)=b, a(t)=a, w(t)=w,\ X(t)=X(\xi)$ for $t\in[0,T],\ \xi\in R$ and random disturbances $e_t(\xi)$ are such that $-\frac{a-b(c_r+c_s)}{2}\leqslant e_t(\xi)\leqslant 2U-\frac{a-b(c_r+c_s)}{2}$, then the supplier's wholesale pricing policy $w^*(t)=\frac{a-b(c_r-c_s)}{2b}$, and the retailer's processing $u^*(t,\xi)=\frac{a+2e_t(\xi)-b(c_r+c_s)}{4}$ and pricing $p^*(t,\xi)=\frac{3a+2e_t(\xi)+b(c_r+c_s)}{4b}$ policies constitute the unique Stackelberg equilibrium for $t\in[0,T]$.*

According to Theorem 2, if random demand disturbances $e_t(\xi)$ are bounded by

$$-\frac{a-b(c_r+c_s)}{2}\leqslant e_t(\xi)\leqslant 2U-\frac{a-b(c_r+c_s)}{2},\tag{18}$$

the optimal supplier's wholesale price $w^*(t)=\frac{a-b(c_r-c_s)}{2b}$ is proportional to the demand potential and the supplier's processing cost. This wholesale price, however, decreases when the retailer's processing cost and the customer sensitivity increase. Naturally, the greater the maximum processing rate, U, the wider the bounds determined by conditions (18), which implies that the steady-state equilibrium of the supply chain can be retained even with sizable disturbances in demand.

4. The transient-state analysis

In this section we assume first that since the promotion time is much shorter than the committed contract period T, the supplier chooses the wholesale price as determined in Theorems 1 and 2 to maintain a steady-state; a new wholesale price can only be selected at a predetermined date for a limited promotional period. In response, the retailer will change his policy accordingly. This changeover induces a transient state in the supply chain in which both the supplier and retailer attempt to use the increased customer sensitivity during the limited promotional period to raise sales.

Secondly, since T is longer than the promotion duration, we assume that the supply chain which is in a steady-state (characterized by demand potential a_1 and sensitivity b_1) at time $t=0$ will return to this state by time $t=T$ after the promotion period, which starts at $t_s>0$ and ends at time $t_f<T$. This implies that the optimality conditions (10)–(13), 16,17 derived in the previous section remain the same, but that $w(t)$ is no longer constant and is defined by Eq. (1), $w_1=w^*(t)=\frac{a-b(c_r-c_s)}{2b}$, where $a=a_1$ and $b=b_1$.

4.1. Deterministic demand

We start off again by considering the case when demand is deterministic, i.e., e_t=0 for any t. To derive the retailer's best response function, we distinguish between two types of transient states: brief and maximal changeover. The difference between the two transient states is due to a temporal steady-state the supply chain may reach during the promotion. The presence of this temporal steady-state implies that the retailer has enough time to optimally reduce prices to a minimum level corresponding to the promotional wholesale price w_2. This phenomenon can be viewed as the maximum effect that a promotional initiative can cause, which is why in this paper we focus on this type of transient state, as discussed in the following lemma.

Lemma 3. *Let* $a(t) - b(t)(c_r + w(t)) \geqslant 0$, $d^* = \frac{a_1 - b_1(c_r + w_1)}{2}$, $d^{**} = \frac{a_2 - b_2(c_r + w_2)}{2}$, $w_1 > w_2$. *If* $t_1 < t_s$, $t_2 > t_s$, $t_3 < t_f$, $t_4 > t_f$, $t_2 \leqslant t_3$ *satisfy the following equations*:

$$\begin{aligned} U(t_2 - t_s) &= \frac{1}{2}(a_1(t_s - t_1) + a_2(t_2 - t_s)) - \frac{1}{2}(b_1(t_s - t_1) + b_2(t_2 - t_s))(c_r + w_1 + h^- t_1) \\ &\quad + \frac{1}{4}h^-(b_1(t_s^2 - t_1^2) + b_2(t_2^2 - t_s^2)), \quad h^-(t_2 - t_1) = w_1 - w_2, \end{aligned} \tag{19}$$

$$\begin{aligned} U(t_f - t_3) &= \frac{1}{2}(a_1(t_4 - t_f) + a_2(t_f - t_3)) - \frac{1}{2}(b_1(t_4 - t_f) + b_2(t_f - t_3))(c_r + w_2 - h^+ t_3) \\ &\quad - \frac{1}{4}h^+(b_1(t_4^2 - t_f^2) + b_2(t_f^2 - t_3^2)), \quad h^+(t_4 - t_3) = w_1 - w_2, \end{aligned} \tag{20}$$

then $X(t) = 0$ *for* $0 \leqslant t \leqslant t_1$, $t_2 \leqslant t \leqslant t_3$, $t_4 \leqslant t \leqslant T$; $X(t) < 0$ *for* $t_1 < t < t_2$, $X(t) > 0$ *for* $t_3 < t < t_4$; *the optimal retailer's processing policy is*

$$u(t) = d^* \quad \text{for } 0 \leqslant t < t_1 \quad \text{and} \quad t_4 \leqslant t \leqslant T, \qquad u(t) = d^{**} \quad \text{for } t_2 \leqslant t < t_3,$$
$$u(t) = U \quad \text{for } t_s \leqslant t < t_2 \quad \text{and} \quad t_3 \leqslant t < t_f, \qquad u(t) = 0 \quad \text{for } t_1 \leqslant t < t_s \quad \text{and} \quad t_f \leqslant t < t_4;$$

and the optimal retailer's pricing policy is

$$p(t) = \frac{a(t) + b(t)(c_r + w_1 - h^-(t - t_1))}{2b(t)} \quad \text{for } t_1 \leqslant t < t_2, \quad p(t) = \frac{a_2 + b_2(c_r + w_2)}{2b_2}$$
$$\text{for } t_2 \leqslant t < t_3, \quad p(t) = \frac{a_1 + b_1(c_r + w_1)}{2b_1} \quad \text{for } 0 \leqslant t < t_1, \ t_4 \leqslant t \leqslant T,$$
$$p(t) = \frac{a(t) + b(t)(c_r + w_2 + h^+(t - t_3))}{2b(t)} \quad \text{for } t_3 \leqslant t < t_4.$$

The optimal solution derived in Lemma 3 is illustrated in Fig. 1. According to this solution, it is beneficial for the retailer to change pricing and processing policies in response to a reduced wholesale price and increased customer price sensitivity during the promotion. The change is characterized by instantaneous jumps upward in quantities ordered and downward in retailer prices at the point the promotion starts and vice versa at the point the promotion ends. Inventory surplus by the end of the promotion indicates that the retailer orders more goods during the promotional period than it is able to sell (forward buying). Moreover, the retailer starts to lower prices even before the promotion starts. This strategy allows building greater demands by the beginning of the promotion period and taking advantage of the reduced wholesale price during the promotion. This is accomplished gradually so that a trade-off between the inventory backlog (surplus) cost and the wholesale price is sustained over time. Fig. 1 shows that any reduction in wholesale price results first in backlogs and then surplus inventories. This is in contrast to a steady-state with no inventories being held. In addition, the total retailer's order quantity increases with the decrease of the wholesale price as formulated in the following corollary.

Corollary 1. *If* $a(t) - b(t)(c_r + w(t)) \geqslant 0$, *the lower the promotional wholesale price*, w_2, *the greater the total order* $\int_{t_1}^{t_4} u(t)\,dt = \int_{t_1}^{t_4} d(t)\,dt$ *and the lower the overall product pricing* $\int_{t_1}^{t_4} p(t)\,dt$.

Note that the retailer's problem is convex, but not strictly so, since it involves piece-wise linear terms in objective function (4). This implies that in contrast to Lemmas 1 and 2, where state and co-state variables were uniquely fixed at a constant level, there could be multiple solutions in the transient system that provide the same optimal value for the objective function. The following lemma resolves this ambiguity.

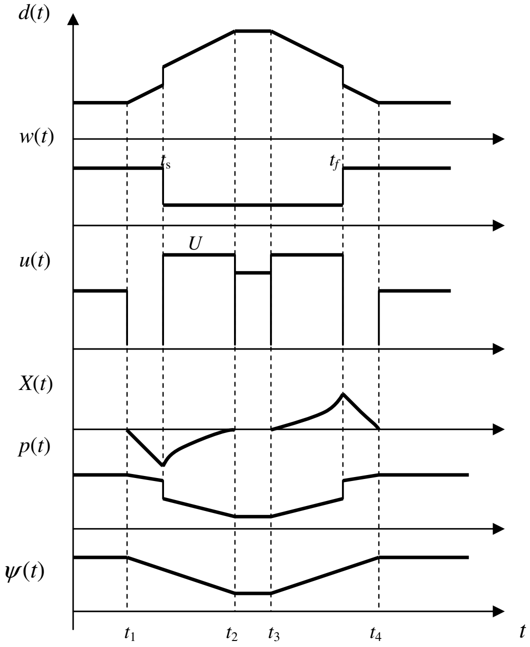

Fig. 1. Optimal retailer policies under promotion (the case of symmetric costs, $h^+ = h^-$).

Lemma 4. *The optimal retailer's response determined by Lemma 3 is unique.*

Lemmas 3 and 4 identify a unique retailer's strategy in the presence of a transient state during a promotion period. To define the corresponding supplier's strategy over interval $[0, T]$, we substitute the best retailer's response from Lemma 3 into objective function (2):

$$E\left[\int_0^T [w(t)u(t) - c_s u(t)]\,\mathrm{d}t\right] = \int_0^{t_1} (w_1 - c_s)d^*\,\mathrm{d}t + \int_{t_s}^{t_2} (w_2 - c_s)U\,\mathrm{d}t + \int_{t_2}^{t_3} (w_2 - c_s)d^{**}\,\mathrm{d}t + \int_{t_3}^{t_f} (w_2 - c_s)U\,\mathrm{d}t + \int_{t_4}^{T} (w_1 - c_s)d^*\,\mathrm{d}t.$$

That is

$$E\left[\int_0^T [w(t)u(t) - c_s u(t)]\,\mathrm{d}t\right] = (w_1 - c_s)d^*(T - t_4 + t_1) + (w_2 - c_s)U(t_2 - t_s + t_f - t_3) + (w_2 - c_s)d^{**}(t_3 - t_2). \tag{21}$$

Applying the first order optimality conditions to this function with respect to w_2 and denoting the result by $F(w_2)$, we obtain

$$F(w_2) = d^*(w_1 - c_s)[t_1 - t_4]'_{w_2} + U(t_f - t_s) + U[(w_2 - c_s)(t_2 - t_3)]'_{w_2} + [d^{**}(w_2 - c_s)(t_3 - t_2)]'_{w_2} = 0. \tag{22}$$

To show the uniqueness of the equilibrium for a transient state, we need the following property.

Lemma 5. *Let* $R_1 = -A_1^* - A_2^* + (w_1 - w_2)(\frac{1}{h^-} + \frac{1}{h^+})$, $a(t) - b(t)(c_r + w(t)) \geqslant 0$, $b_2 > b_1$,

$$R_2 = \frac{U(t_3 - t_2) - d^*(w_1 - c_s)[t_1 - t_4]'_{w_2} - (U - \hat{d})(w_1 - c_s)\left([(t_1 - t_4)]'_{w_2} - \left[\frac{1}{h^-} + \frac{1}{h^+}\right]\right) - (\hat{d} - \frac{b_2}{2}(w_1 - c_s))(t_3 - t_2)}{U},$$

and $\hat{d} = \frac{a_2 - b_2(c_r + w_1)}{2}$, *where* A_1^* *and* A_2^* *are determined by the solutions of (19) and (20) (see Appendix A). If* $R_1 \leqslant t_f - t_s \leqslant R_2$, *then Eq. (22) has only one root* α, *such that* $c_s < \alpha < w_1 = \frac{a_1 - b_1(c_r - c_s)}{2b_1}$.

The optimal supplier's wholesale price versus retailer's price and order quantity under a limited promotion period is presented in the following theorem.

Theorem 3. *If $a_1 - b_1(c_r + c_s) \geqslant 0$, $a_2 - b_2(c_r + \alpha) \geqslant 0$, and $R_1 \leqslant t_f - t_s \leqslant R_2$, then the supplier's wholesale pricing policy $w^*(t) = w_1 = \frac{a_1 - b_1(c_r - c_s)}{2b_1}$ for $0 \leqslant t < t_s$, $t_f \leqslant t \leqslant T$ and $w^*(t) = w_2 = \alpha$ for $t_s \leqslant t < t_f$, and the retailer's processing $u^*(t)$ and pricing $p^*(t)$ policies, determined by Lemma 3, constitute the unique Stackelberg equilibrium for $t \in [0, T]$.*

The existence of equilibrium wholesale price $w^*(t) = w_2 = \alpha$ stated in Theorem 3 readily results in the following corollary.

Corollary 2. *Let $a_1 - b_1(c_r + c_s) \geqslant 0$, $a_2 - b_2(c_r + \alpha) \geqslant 0$, and $R_1 \leqslant t_f - t_s \leqslant R_2$, if the customer sensitivity increases during the promotion period, $b_2 > b_1$, then the wholesale price decreases $w_2 < w_1$.*

From Corollaries 1 and 2, it immediately follows that during higher demand the retail price falls (Corollary 1) when customer sensitivity increases (Corollary 2). Moreover, the retailer starts to lower prices even before the promotion starts (Lemma 3). This phenomenon has been widely observed in empirical studies of retail prices during and close to holidays (see, for example, Chevalier et al., 2003; Bils, 1989; Warner and Barsky, 1995). Furthermore, for the linear relationship between customer sensitivity and demand potential, $a(t) = g(t) + b(t)P$, discussed in the introduction, we can estimate the minimum total order quantity increase in a transient state. This is shown in the following lemma.

Lemma 6. *If $a_1 = g + b_1P$, $a_2 = g + b_2P$ and $P > \frac{1}{2}(\frac{a_1}{b_1} + c_r + c_s)$, then the total order quantity in the transient state, $[t_1, t_4]$, exceeds the total order quantity in the steady-state of the same duration, $[t_1, t_4]$, by more than $\frac{1}{2}\left(P - \frac{1}{2}(\frac{a_1}{b_1} + c_r + c_s)(b_2 - b_1)\right)(t_f - t_s)$.*

Note, that $P > \frac{1}{2}(\frac{a_1}{b_1} + c_r + c_s)$ implies $P > \frac{g}{b_1} + c_r + c_s$, i.e., this condition of Lemma 6 may hold even if the price, $p^*(t)$, at the steady-state (see Theorem 1) is above the list price P. An immediate corollary is in order.

Corollary 3. *If $a_1 = g + b_1P$, $a_2 = g + b_2P$ and $P > \frac{1}{2}(\frac{a_1}{b_1} + c_r + c_s)$, then the stronger the increase in customer sensitivity,b_2, the greater the increase of the total order quantity during the transient state compared to the steady-state.*

4.2. The effect of random demand disturbance

The sensitivity of the optimal solution determined in Theorem 3 to a bounded change in demand is different from that for a steady-state. This is due to the fact that instead of constant inventories maintained at a steady-state, the retailer's best response during a promotion is characterized by inventory levels changing over time as shown in the following lemma.

Lemma 7. *If the random disturbances $e_t(\xi)$ are such that $a(t) + e_t(\xi) - b(t)(c_r + w(t)) \geqslant 0$, $t \in [0, T]$, $a(t) + e_t(\xi) - b(t)(c_r + w(t)) \leqslant 2U$ for $0 \leqslant t \leqslant t_1$, $t_2 \leqslant t \leqslant t_3$, $t_4 \leqslant t \leqslant T$, $\xi \in R$;*

$$-\int_{t_1}^{t}(a(\tau) + e_\tau(\xi) - b(\tau)p(\tau,\xi))\mathrm{d}\tau + U(t - t_s) \leqslant 0, \quad t_1 < t < t_2;$$

$$U(t - t_3) - \int_{t_3}^{t}(a(\tau) + e_\tau(\xi) - b(\tau)p(\tau,\xi))\mathrm{d}t \geqslant 0, \quad t_3 < t < t_4;$$

$$\int_{t_1}^{t_2} e_t(\xi)\mathrm{d}t = 0, \quad \int_{t_3}^{t_4} e_t(\xi)\mathrm{d}t = 0, \quad \textit{then } X(t,\xi) = 0 \quad \textit{for } 0 \leqslant t \leqslant t_1,\ t_2 \leqslant t \leqslant t_3,\ t_4 \leqslant t \leqslant T;$$

$$X(t,\xi) < 0 \quad \textit{for } t_1 < t < t_2, \quad X(t,\xi) > 0 \quad \textit{for } t_3 < t < t_4;$$

the optimal retailer's processing policy is $u(t,\xi) = \frac{a_1 + e_t(\xi) - b_1(c_r + w_1)}{2}$ for $0 \leqslant t < t_1$ and $t_4 \leqslant t \leqslant T$, $u(t,\xi) = \frac{a_2 + e_t(\xi) - b_2(c_r + w_2)}{2}$ for $t_2 \leqslant t < t_3$, $u(t,\xi) = U$ for $t_s \leqslant t < t_2$ and $t_3 \leqslant t < t_f$, $u(t,\xi) = 0$ for $t_1 \leqslant t < t_s$ and $t_f \leqslant t < t_4$; and the optimal retailer's pricing policy is $p(t,\xi) = \frac{a(t) + e_t(\xi) + b(t)(c_r + w_1 - h^-(t - t_1))}{2b(t)}$ for $t_1 \leqslant t < t_2$,

$p(t,\xi)=\frac{a_2+e_t(\xi)+b_2(c_r+w_2)}{2b_2}$ *for* $t_2 \leqslant t < t_3$, $p(t,\xi)=\frac{a_1+e_t(\xi)+b_1(c_r+w_1)}{2b_1}$ *for* $0 \leqslant t < t_1$, $t_4 \leqslant t \leqslant T$, $p(t,\xi)=\frac{a(t)+e_t(\xi)+b(t)(c_r+w_2+h^+(t-t_3))}{2b(t)}$ *for* $t_3 \leqslant t < t_4$.

According to Lemma 7, random fluctuations in customer demands will not affect the optimal retailer's policy as long as there is no change in the cumulative demand at the breaking points, t_2 and t_4. As a result, the processing rate remains the same though the prices change and the inventory level may fluctuate substantially between time points t_1 and t_2 as well as t_3 and t_4. Note, though the cumulative conditions imposed on $e_t(\xi)$ in Lemma 7 are restricting, according to the celebrated Ergodic Theorem, if the process of disturbance $e_t = e_k$, for $t^k \leqslant t < t^{k+1}$, $k = 0, 1, 2, \ldots$, is Ergodic, then the smaller the time intervals $[t^k, t^{k+1}]$ (compared to the intervals $[t_1, t_2]$ and $[t_3, t_4]$), the closer the time average to the space average. That is, $\frac{1}{t_2-t_1}\int_{t_1}^{t_2} e_t(\xi)\,\mathrm{d}t \to E[e_k] = 0$ and, thus, the conditions $\int_{t_1}^{t_2} e_t(\xi)\,\mathrm{d}t = \int_{t_3}^{t_4} e_t(\xi)\,\mathrm{d}t = 0$ are likely to hold.

Consequently, the optimal solution derived in Lemma 7 is based on the co-state behavior identical to that of Lemma 3 (see the proof of Lemma 3 in Appendix A). Therefore, using the same arguments as in Lemma 4, we can conclude that this solution is also unique. Furthermore, similar to the retailer's best response when disturbances occur in a steady-state, one can view the optimal solution during the promotion conditions of Lemma 7 as a feedback policy. Indeed, the processing and pricing policies are such that inventory levels are kept at zero when the supply chain is in a new steady-state during the promotion, i.e., for $t_2 \leqslant t < t_3$. On the other hand, the remaining promotion time is characterized by an integral feedback, $\pi^0(X(t), t)$, where the upper index, 0, stands for the critical number $X^* = 0$ (threshold) which the feedback depends on. This is summarized below:

$$u(t,\xi) = \pi_u^0(X(t,\xi),t) = \begin{cases} 0, & \text{if } X(t,\xi) \leqslant 0 \text{ and } t_1 \leqslant t < t_s; \\ U, & \text{if } X(t,\xi) < 0 \text{ and } t \geqslant t_s; \\ U, & \text{if } X(t,\xi) \geqslant 0 \text{ and } t_3 \leqslant t < t_f; \\ 0, & \text{if } X(t,\xi) > 0 \text{ and } t \geqslant t_f; \end{cases}$$

$$p(t,\xi) = \pi_p^0(X(t,\xi),t) = \begin{cases} \dfrac{a(t)+e_t(\xi)+b(t)(c_r+w_1-h^-(t-t_1))}{2b(t)}, & \text{if } X(t) < 0; \\ \dfrac{a(t)+e_t(\xi)+b(t)(c_r+w_2+h^+(t-t_3))}{2b(t)}, & \text{if } X(t) > 0. \end{cases}$$

An important insight of these feedback policies is that the traditional marketing assumption on time invariance of optimal feedback does not necessarily hold if the supply chain has a transient state.

Relying on Lemma 7, we determine the bounds for stochastic disturbances so that the optimal supplier's strategy remains the same as in Theorem 3.

Theorem 4. *If the random disturbances* $e_t(\xi)$ *are such that* $a_1 + e_t(t,\xi) - b_1(c_r + c_s) \geqslant 0$, $a_2 + e_t(t,\xi) - b_2(c_r + \alpha) \geqslant 0$, $t \in [0, T]$, $a_1 + 2e_t(t,\xi) - b_1(c_r + c_s) \leqslant 4U$, *for* $0 \leqslant t \leqslant t_1$ *and* $t_4 \leqslant t \leqslant T$, $a_2 + e_t(t,\xi) - b_2(c_r + \alpha) \leqslant 2U$ *for* $t_2 \leqslant t \leqslant t_3$, $\xi \in R$; $-\int_{t_1}^{t}(a(\tau) + e_\tau(\xi) -> b(\tau)p(\tau,\xi))\,\mathrm{d}\tau + U(t - t_s) \leqslant 0$, $t_1 < t < t_2$, $U(t - t_3) - \int_{t_3}^{t}(a(\tau) + e_\tau(\xi) - b(\tau)p(\tau,\xi))\,\mathrm{d}t \geqslant 0, t_3 < t < t_4$, $\int_{t_1}^{t_2} e_t(\xi)\,\mathrm{d}t = \int_{t_3}^{t_4} e_t(\xi)\,\mathrm{d}t = 0$, *and* $R_1 \leqslant t_f - t_s \leqslant R_2$, *then the supplier's wholesale pricing policy* $w^*(t) = w_1 = \frac{a_1 - b_1(c_r - c_s)}{2b_1}$ *for* $0 \leqslant t < t_s$, $t_f \leqslant t \leqslant T$ *and* $w^*(t) = w_2 = \alpha$ *for* $t_s \leqslant t < t_f$, *and the retailer's processing* $u^*(t,\xi)$ *and pricing* $p^*(t,\xi)$ *policies determined by Lemma 6 are the unique Stackelberg equilibrium for* $t \in [0, T]$.

As shown in Theorems 3 and 4, as well as Corollaries 1–3, the optimal Stackelberg solution implies that if customer sensitivity increases during a promotional period, then both the retailer and the supplier increase their profits compared to a solution which disregards the change in customer sensitivity. This however, does not necessarily mean that profits during the promotion will exceed those gained during regular operation at a steady-state. This is to say, that on special occasions like Christmas, customer sensitivity may increase without any promotional initiative and the chain will have no other option than to respond. On the other hand, if a promotional initiative is assessed in advance as not beneficial in regard to regular profits then it can be abandoned. The necessary and sufficient condition with respect to the profitability of a limited-time, $R_1 \leqslant t_f - t_s \leqslant R_2$, promotion initiated by the leader is straightforwardly obtained from Eq. (21):

Table 1
Wholesale prices w_2^* in transient state and profit gaps between transient and steady-state (10^6\$) of the supplier $\theta_1(b_2)$ and retailer $\theta_2(b_2)$

	$h^+=1,\ h^-=2,\ c_r=30,\ c_s=60$	$h^+=1,\ h^-=10,\ c_r=30,\ c_s=60$	$h^+=1,\ h^-=2,\ c_r=60,\ c_s=30$
$b_2=12$–24			
$\theta_1(b_2)(\theta_2(b_2))$	–	–	–
w_2^*	No equilibrium	No equilibrium	No equilibrium
$b_2=28$			
$\theta_1(b_2)(\theta_2(b_2))$	4.2342 (1.8058)	4.2540 (1.8669)	4.2342 (1.8058)
w_2^*	125.6560	125.2240	95.6560
$b_2=32$			
$\theta_1(b_2)(\theta_2(b_2))$	0.6568 (0.5914)	0.7292 (0.5289)	0.6568 (0.5914)
w_2^*	114.0560	113.2240	84.0560
$b_2=36$			
$\theta_1(b_2)(\theta_2(b_2))$	−2.0835 (0.0371)	−1.9369 (−0.3428)	−2.0835 (0.0371)
w_2^*	104.5200	103.3680	74.5200
$b_2=40$			
$\theta_1(b_2)(\theta_2(b_2))$	−4.1982 (−0.0935)	−3.9647 (-1.1398)	−4.1982 (−0.0935)
w_2^*	96.5756	95.1680	66.5756
$b_2=44$			
$\theta_1(b_2)(\theta_2(b_2))$	−5.8350 (−0.0398)	−5.5084 (-2.1196)	−5.8350 (-0.0398)
w_2^*	89.8640	88.2800	58.8640
$b_2=48$			
$\theta_1(b_2)(\theta_2(b_2))$	−7.0995 (0.1026)	−6.6780 (−3.0038)	−7.0995 (0.1026)
w_2^*	84.1444	82.4160	54.1444
$b_2=52$			
$\theta_1(b_2)(\theta_2(b_2))$	−8.0689 (0.3553)	−7.5535 (−3.6440)	−8.0689 (0.3553)
w_2^*	79.2080	77.3840	49.2080

If $\theta_1(b_2) = (w_2 - c_s)U(t_2 - t_s + t_f - t_3) + (w_2 - c_s)d^{**}(t_3 - t_2) - (w_1 - c_s)d^*(t_4 - t_1) > 0$, then the supplier (the leader) will gain from the promotion an extra profit compared to the regular (steady-state) profits for the same period of time. Similarly, from (4) one can define a gap function, $\theta_2(b_2)$, so that the retailer would have an extra profit if $\theta_2(b_2 > 0)$. Since these conditions involve extremely large expressions of the switching time points, we illustrate the evolution of profit gaps $\theta_1(b_2)$ and $\theta_2(b_2)$ quantitatively for different customer sensitivities and fixed promotion times. The interpretation is immediate – when both gaps are positive, the promotion is beneficial for both the leader and the follower. Calculations for $U = 10{,}000$, $a_1 = 2500$, $a_2 = 6000$ product units per time unit, $b_1 = 10$ product units per dollar and time unit, $t_s = 100$, $t_f = 300$ and $T = 1000$ time units are presented in Table 1.

From Table 1, we see that there is a bounded interval to the customer sensitivity values b_2 for which an equilibrium exists. The existence of the equilibrium starts from $b_2 > 24$ which ensures our general assumption on an increase in demand elasticity, $\frac{a_1}{b_1} > \frac{a_2}{b_2}$, and terminates at $b_2 > 52$ when the condition, $a_2 - b_2(c_r + w_2) \geqslant 0$, of Theorem 3 no longer holds. More importantly, the range of values is such that the promotion gains extra profits for both the supplier and retailer (i.e., gaps $\theta_1(b_2)$ and $\theta_2(b_2)$ are both positive) from $b_2 = 28$ to $b_2 = 32$. This result is due to a non-linear relationship between the demand potential, a_2, which remains the same and sensitivity, b_2, which increases. The profitability range could be extended if a linear relationship, $a(t) = g + b(t)P$ (see Lemma 6), were used in the example. Under such conditions, a_2 would always increase with b_2.

5. Conclusion

Though a steady-state Stackelberg equilibrium of the supply chain exists, this is not always the case when the chain is in a promotion state. The main reason for this is an instantaneous increase in customer sensitivity during a predetermined period of time. The retailer's equilibrium strategy is to purchase more goods during the promotional period than the retailer expects to sell. Specifically, it builds some backlog demand around the starting point of the promotion and some inventory around the ending point of the promotion, and keeps zero inventory elsewhere. We show that both wholesale and product prices must be decreased in response to the change in customer sensitivity and that there is an upper time bound for such a promotion to attain an equilibrium. Moreover, this equilibrium is not necessarily beneficial in comparison to regular, steady-state sales. A necessary and sufficient condition is suggested which can be used in two alternative ways. One is to find the promotion time limit so that it remains profitable for a given customer sensitivity. This, of course, is possible when the chain members initiate the promotion and the promotion time is controllable. The other way is to find the range of customer sensitivity levels such that the promotion is beneficial during a given promotion time. This case corresponds to the sales during special dates which are not controllable and is illustrated with an example.

Finally, we considered classic contract conditions characterized by the supplier's commitment to a constant wholesale price if the retailer purchases a certain amount of products for a period of time. However, if the supplier is not fully committed, then the Stackelberg strategy is not necessary time-consistent. Indeed, we feel that further exploration into the development of coordinating contracts and incentives to ensure that the equilibrium wholesale price remains time-consistent during a promotion is an important direction for future research. This naturally complements, of course, research into the challenges posed by the competitive multi-supplier/retailer setting.

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Appendix A

A.1. Stochastic optimality conditions for the retailer's problem

We derive optimal processing rate $u(t,\xi)$ and pricing $p(t,\xi)$ by applying a needle, ε, control variations $\delta u(t,\xi)$ and $\delta p(t,\xi)$ for the realization ξ at point t:

$$\delta u(\tau,\xi)=\begin{cases}\delta u, & \text{if } t\leqslant\tau\leqslant t+\varepsilon,\\ 0, & \text{otherwise,}\end{cases}\qquad \delta p(\tau,\xi)=\begin{cases}\delta p, & \text{if } t\leqslant\tau\leqslant t+\varepsilon,\\ 0, & \text{otherwise.}\end{cases}\tag{A.1}$$

Then if two realizations coincide up to point t, the processing and pricing rates must coincide until this point, i.e., the non-anticipativity constraint is

$$u(\tau,\xi)=u(\tau,\xi'),\quad p(\tau,\xi)=p(\tau,\xi')\quad \text{for all } \xi'\in R(t,\xi),\ 0\leqslant\tau\leqslant t.\tag{A.2}$$

This, by taking into account (A.1), results in

$$\delta u(\tau,\xi)=\delta u(\tau,\xi'),\quad \delta p(\tau,\xi)=\delta p(\tau,\xi')\quad \text{for all } \xi'\in R(t,\xi),\ 0\leqslant\tau\leqslant T.\tag{A.3}$$

Consequently, the influence of variation (A.1)–(A.3) on the inventory level $X(t,\xi)$ in the first order of ε is

$$\delta X(\tau,\xi')=\begin{cases}\varepsilon\delta u+b(t)\varepsilon\delta p, & \text{if } \tau>t\\ 0, & \text{otherwise}\end{cases}\qquad \text{for all } \xi'\in R(t,\xi).\tag{A.4}$$

With respect to (A.1)–(A.4), variation of the objective function (4) results in

$$\delta J_{\mathrm{r}} = (a(t) - 2b(t)p(t,\xi) + e_t(\xi))\varepsilon\delta p - c_{\mathrm{r}}\varepsilon\delta u - w(t)\varepsilon\delta u - E_{R(t,\xi)}\left[\int_t^T \frac{\mathrm{d}h(X(\tau,\xi))}{\mathrm{d}X(\tau,\xi)}(\varepsilon\delta u + b(t)\varepsilon\delta p)\,\mathrm{d}\tau\right]. \tag{A.5}$$

Introducing a co-state variable as defined by (10), we obtain

$$\delta J_{\mathrm{r}} = (a(t) - 2bp(t,\xi) + e_t(\xi))\varepsilon\delta p - c_{\mathrm{r}}\varepsilon\delta u - w(t)\varepsilon\delta u + E_{R(t,\xi)}[\psi(t,\xi)](\varepsilon\delta u + b\varepsilon\delta p). \tag{A.6}$$

Requiring that no variation of the objective function can improve it, $\delta J_{\mathrm{r}} \leqslant 0$, and considering only price-dependent terms of variation (A.6)

$$\delta J_{\mathrm{r}}(p) = (a(t) - 2b(t)p(t,\xi) + e_t(\xi))\varepsilon\delta p + E_{R(t,\xi)}[\psi(t,\xi)]b(t)\varepsilon\delta p \leqslant 0,$$

we find that when the price is maximal, $p(t,\xi) = a(t)/b(t)$, a possible variation δp can only be negative, therefore, $(a(t) - 2b(t)p(t,\xi) + e_t(\xi)) + E_{R(t,\xi)}[\psi(t,\xi)]b(t) \geqslant 0$. That is, $a(t) + e_t(\xi) + E_{R(t,\xi)}[\psi(t,\xi)]b(t) > 2a(t)$. On the other hand, when the price is minimal, $p(t,\xi) = 0$, the only possible variation is positive and, therefore, $a(t) + e_t(\xi) + E_{R(t,\xi)}[\psi(t,\xi)]b(t) < 0$. An intermediate pricing regime is derived when $0 \leqslant a(t) + e_t(\xi) + E_{R(t,\xi)}[\psi(t,\xi)]b(t) \leqslant 2a(t)$ from $(a(t) - 2b(t)p(t,\xi) + e_t(\xi)) + E_{R(t,\xi)}[\psi(t,\xi)]b(t) = 0$. These stochastic optimality conditions are summarized in (16).

Similarly, considering only processing-rate-dependent terms of variation (A.6)

$$\delta J_{\mathrm{r}} = -c_{\mathrm{r}}\varepsilon\delta u - w(t)\varepsilon\delta u + E_{R(t,\xi)}[\psi(t,\xi)]\varepsilon\delta u \leqslant 0,$$

we find that when $u(t,\xi) = U$, only negative variation is feasible, $\delta u < 0$, i.e., $\delta J_{\mathrm{r}} \leqslant 0$, if $E_{R(t,\xi)}[\psi(t,\xi)] \geqslant c_{\mathrm{r}} + w(t)$. Similarly, if $u(t,\xi) = 0$, $\delta u > 0$ and $E_{R(t,\xi)}[\psi(t,\xi)] \leqslant c_{\mathrm{r}} + w(t)$. Finally, if $0 < u(t,\xi) < U$, $E_{R(t,\xi)}[\psi(t,\xi)] = c_{\mathrm{r}} + w(t)$. To eliminate the ambiguity of the last condition, we differentiate it over an interval of time. Then by taking into account (1) and (10), we find that $X(t,\xi) = 0$ over this interval, which, with respect to (5), results in $u(t,\xi) = a(t) + e_t(\xi) - b(t)p(t,\xi)$, if $E_{R(t,\xi)}[\psi(t,\xi)] = c_{\mathrm{r}} + w(t)$, as summarized in (17).

A.2. Solution of Eq. (19)

$$t_1^* = t_{\mathrm{s}} - A_1^* \quad \text{and} \quad t_2^* = t_1^* + f_1,$$

where

$$f_1 = \frac{w_1 - w_2}{h^-}, \quad f_2 = c_{\mathrm{r}} + w_1, \quad A_1^* = \frac{U + \frac{1}{2}a_1 - \frac{1}{2}a_2 - \frac{1}{2}b_1 f_2 + \frac{1}{2}b_2 f + [D_1^*]^{\frac{1}{2}}}{\frac{1}{2}h^-[b_2 - b_1]}$$

$$D_1^* = U^2 + Ua_1 - Ua_2 - Ub_1f_2 + Ub_2f_2 + \frac{1}{4}a_1^2 + \frac{1}{4}a_2^2 - \frac{1}{2}a_1a_2 - \frac{1}{2}a_1b_1f_2 + \frac{1}{2}a_2b_1f_2 - \frac{1}{2}a_2b_2f_2 + \frac{1}{2}a_1b_2f_2 + \frac{1}{4}b_1^2f_2^2 - \frac{1}{2}b_1b_2f_2^2 + \frac{1}{4}b_2^2f_2^2 - \left(\frac{1}{2}h^-b_1a_2f_1 - \frac{1}{2}h^-b_1b_2f_1f_2 + \frac{1}{4}h^{-2}b_1b_2f_1^2 - h^-b_1Uf_1 - \frac{1}{2}h^-b_2a_2f_1 + \frac{1}{2}h^-b_2^2f_1f_2 - \frac{1}{4}h^{-2}b_2^2f_1^2 + h^-b_2Uf_1\right).$$

A.3. Solution of Eq. (20)

$$t_4^* = t_{\mathrm{f}} + A_2^* \quad \text{and} \quad t_3^* = t_4^* - f_1, \quad f_1 = \frac{w_1 - w_2}{h^+},$$

where

$$f_2 = c_{\mathrm{r}} + w_1, \quad A_2^* = \frac{U + \frac{1}{2}a_1 - \frac{1}{2}a_2 - \frac{1}{2}b_1f_2 + \frac{1}{2}b_2f_2 - \frac{1}{2}b_1h^+f_1 + \frac{1}{2}b_2f_1h^+ + [D_2^*]^{\frac{1}{2}}}{\frac{1}{2}h^+[b_2 - b_1]},$$

$$D_2^* = U^2 + Ua_1 - Ua_2 + \frac{1}{4}a_1^2 + \frac{1}{4}a_2^2 - \frac{1}{2}a_1a_2 - b_1f_2U + b_2f_2U - \frac{1}{2}a_1b_1f_2 + \frac{1}{2}a_1b_2f_2 + \frac{1}{2}a_2b_1f_2 - \frac{1}{2}a_2b_2f_2 - b_1h^+f_1U$$
$$+ b_2h^+f_1U + \frac{1}{4}b_1^2f_2^2 + \frac{1}{4}b_2^2f_2^2 - \frac{1}{2}a_1b_1f_1h^+ - \frac{1}{2}a_2b_2f_1h^+ + \frac{1}{2}a_1b_2f_1h^+ + \frac{1}{2}a_2b_1f_1h^+ + \frac{1}{2}b_2^2f_1f_2h^+ - \frac{1}{2}b_1b_2f_1^2h^{+2}$$
$$- \frac{1}{2}b_1b_2f_2^2 + \frac{1}{2}b_1^2f_1f_2h^+ - b_1b_2f_1f_2h^+ + \frac{1}{4}b_1^2f_1^2h^{+2} + \frac{1}{4}b_2^2f_1^2h^{+2}$$
$$- \left(-b_1h^+Uf_1 + \frac{1}{2}b_1h^+a_2f_1 - \frac{1}{2}b_1h^+b_2f_1f_2 - \frac{1}{4}b_1h^{+2}b_2f_1^2 + b_2h^+Uf_1 - \frac{1}{2}b_2h^+a_2f_1 + \frac{1}{2}b_2^2h^+f_1f_2 + \frac{1}{4}b_2^2h^{+2}f_1^2\right).$$

Proof of Lemma 1. Consider the following solution for the state, co-state and decision variables:

$$X(t) = 0, \quad \psi(t) = c_r + w, \quad p(t) = \frac{a + b(c_r + w)}{2b} \quad \text{and} \quad u(t) = \frac{a - b(c_r + w)}{2} \quad \text{for } t \in \tau.$$

It is easy to observe that this solution satisfies the optimality conditions (11)–(13). Furthermore, this solution is always feasible if conditions (7) and (14) hold which is ensured by $0 \leqslant a - b(c_r + w) \leqslant 2U$ stated in the lemma. Finally, the retailer's objective function (4) involves only concave and piece-wise linear terms, which implies that the maximum-principle-based optimality conditions are not only necessary, but also sufficient. □

Proof of Theorem 1. Function (15) is concave in w, therefore the first order optimality condition applied to it results in a unique optimal solution $w^*(t) = \frac{a - b(c_r - c_s)}{2b}$ which is feasible if $\frac{a}{b} \geqslant c_r + c_s$, as stated in the theorem. Furthermore, $p^*(t)$ is feasible (meets (7)) due to the same condition, $\frac{a}{b} \geqslant c_r + c_s$. Finally, $u^*(t)$ is feasible if the condition, $0 \leqslant a - b(c_r + w) \leqslant 2U$, stated in lemma 1 holds. Substitution of $w^*(t)$ into this condition as well as into equations for $p(t)$ and $u(t)$ determined in Lemma 1 completes the proof. □

Proof of Lemma 2. Since $0 \leqslant a + e_t(\xi) - b(c_r + w) \leqslant 2U$, it is still feasible to maintain zero inventory level, so that $0 \leqslant u(t,\xi) = \frac{a + e_t(\xi) - b(c_r + w)}{2} \leqslant U$ and $p(t,\xi) = \frac{a + e_t(\xi) + b(c_r + w)}{2b}$ satisfies (7). Then with respect to (10), the co-state variable can remain constant at $c_r + w$ and, therefore, $E_{R(t,\xi)}[\psi(t,\xi)] = c_r + w$. Thus, similar to Lemma 1, the solution stated in Lemma 2 meets the stochastic optimality conditions (16) and (17). □

Proof of Theorem 2. The proof is straightforward. Indeed, calculating the expectation in the objective function (2) and taking into account that $E[e_t] = 0$ we obtain (15) again:

$$E\left[\int_0^T [w(t)u(t) - c_s u(t)]\,dt\right] = E_R\left[\int_0^T \frac{a + e_t(\xi) - b(c_r + w)}{2}(w - c_s)\,dt\right] = \frac{a - b(c_r + w)}{2}(w - c_s)T,$$

which is a concave function in w. Therefore, regardless of the length T, the first order optimality condition leads to the same optimal solution $w^*(t)$ as in Theorem 1, which is always feasible (and greater than c_s) as $\frac{a}{b} \geqslant c_r + c_s$. Finally, one can readily verify that $p^*(t,\xi)$ and $u^*(t,\xi)$ are feasible if $0 \leqslant u^*(t,\xi) \leqslant U$, i.e.,

$$-\frac{a - b(c_r + c_s)}{2} \leqslant e_t(\xi) \leqslant 2U - \frac{a - b(c_r + c_s)}{2},$$

as stated in both Lemma 2 and Theorem 2. □

Proof of Lemma 3. First note, that as mentioned in Lemma 1, the retailer's problem is concave, which implies that the necessary optimality conditions are sufficient.

Consider a solution which is characterized by four breaking points, t_1, t_2, t_3 and t_4 so that the retailer is in a steady-state between time points $t = 0$ and $t = t_1$, between $t = t_2$ and $t = t_3$, and between $t = t_4$ and $t = T$, as described below:

$$X(t) = 0 \quad \text{for } 0 \leqslant t \leqslant t_1,\ t_2 \leqslant t \leqslant t_3 \text{ and } t_4 \leqslant t \leqslant T; \tag{A.7}$$

$$u(t) = d^* \quad \text{for } 0 \leqslant t < t_1 \text{ and } t_4 \leqslant t \leqslant T, \quad u(t) = d^{**} \quad \text{for } t_2 \leqslant t < t_3, \tag{A.8}$$

$$u(t) = U \quad \text{for } t_s \leqslant t < t_2 \text{ and } t_3 \leqslant t < t_f, \quad u(t) = 0 \quad \text{for } t_1 \leqslant t < t_s \text{ and } t_f \leqslant t < t_4; \tag{A.9}$$

$$\psi(t) = c_r + w_1 \quad \text{for } 0 \leqslant t < t_1 \text{ and } t_4 \leqslant t \leqslant T, \quad \psi(t) = c_r + w_2 \quad \text{for } t_2 \leqslant t < t_3 \tag{A.10}$$

$$\psi(t) = c_r + w_1 - h^-(t - t_1) \quad \text{for } t_1 \leqslant t < t_2, \qquad \psi(t) = c_r + w_2 + h^+(t - t_3) \quad \text{for } t_3 \leqslant t < t_4. \tag{A.11}$$

It is easy to observe that the solution (A.7)–(A.11) meets optimality conditions (13) if $a(t) - b(t)(c_r + w(t)) \geqslant 0$, $a(t) \leqslant U$ and there is sufficient time to reach a steady-state during the promotion period, i.e., $t_2 \leqslant t_3$. Furthermore, the optimal pricing policy is immediately derived by substituting the co-state solution (A.10) and (A.11) into $p(t) = \frac{a(t)+b(t)\psi(t)}{2b(t)}$ (see optimality conditions (12)), as stated in the lemma. In turn, this solution is feasible if $p(t) \geqslant 0$, which, with respect to (3) always holds, and $p(t) \leqslant \frac{a(t)}{b(t)}$ (see constraint (7)) or the same $d(t) \geqslant 0$ which always holds as well because, $\frac{a_1}{b_1} > \frac{a_2}{b_2}$, $p(t) \leqslant \frac{a_1+b_1(c_r+w_1)}{2b_1}$ and $w_1 = w^*(t) = \frac{a_1-b_1(c_r-c_s)}{2b_1}$.

Thus, to complete the proof, we need to find the four breaking points and impose that $t_2 \leqslant t_3$. Points t_1 and t_2, are found by solving a system of two equations (A.11) and (A.7) . Specifically, from (A.7) and (5) we find that

$$X(t_2) = -\int_{t_1}^{t_2} (a(t) - b(t)p(t))\,dt + U(t_2 - t_s) = 0. \tag{A.12}$$

By substituting found $p(t)$ into (A.12) we obtain

$$U(t_2 - t_s) = \frac{1}{2}(a_1(t_s - t_1) + a_2(t_2 - t_s)) - \frac{1}{2}(b_1(t_s - t_1) + b_2(t_2 - t_s))(c_r + w_1 + h^- t_1) + \frac{1}{4}h^-(b_1(t_s^2 - t_1^2) + b_2(t_2^2 - t_s^2)),$$

which along with

$$c_r + w_2 = c_r + w_1 - h^-(t_2 - t_1)$$

from (A.10) and (A.11) results in the system of two Eq. (19) in unknown t_1 and t_2 as stated in the lemma.

Similarly, $X(t_4) = U(t_f - t_3) - \int_{t_3}^{t_4}(a(t) - b(t)p(t))\,dt = 0$, which results in

$$U(t_f - t_3) = \frac{1}{2}(a_1(t_4 - t_f) + a_2(t_f - t_3)) - \frac{1}{2}(b_1(t_4 - t_f) + b_2(t_f - t_3))(c_r + w_2 - h^+ t_3) - \frac{1}{4}h^+(b_1(t_4^2 - t_f^2) + b_2(t_f^2 - t_3^2)). \tag{A.13}$$

Considering (A.13) simultaneously with equation

$$c_r + w_2 + h^+(t_4 - t_3) = c_r + w_1$$

from (A.10) and (A.11) results in two Eq. (20) for t_3 and t_4 stated in the lemma. □

Proof of Lemma 4. In contrast to Lemma 1, where state and co-state variables were uniquely fixed at a constant level, the presence of a transient state between steady-states implies only that $X(0) = X(T)$ and $\psi(0) = \psi(T)$, as determined by (11). Thus, one may assume that there may be an optimal solution, different from that derived in Lemma 3, which provides the same optimal value for the objective function (2) and meets (11). In what follows we show by contradiction that there is no other optimal solution than that derived in Lemma 3. Specifically, there could be two alternative solutions for the co-state variable which at steady-state is $\psi(t) = c_r + w_1$, as defined by Lemma 1. In the first solution, the steady-state could be followed by an increase of $\psi(t)$, rather than a decrease as described in Lemma 3. With respect to the optimality conditions (12) and (13), this implies that the retailer orders at maximum rate U and sets increasing prices, which in turn decreases the demand. Since $X(t) = 0$ at a steady-state, this change immediately results in positive inventories (see Eq. (5)). Then according to (10) $\dot{\psi}(t) = h^+ > 0$ until the end of the considered period T, i.e., condition (11) will never be met.

The other alternative is characterized by decreasing $\psi(t)$ (as in Lemma 3), but not entering a new steady-state, $\psi(t) = c_r + w_2$. Then according to (12) and (13), the retailer's price should further decrease and no products should be ordered after point t_2. This implies that the demand increases and the inventory level falls, becoming more and more negative. With respect to (10), this means that $\dot{\psi}(t) = -h^- < 0$, until the end of the considered period T which again contradicts condition (11). □

Proof of Lemma 5. First note that function (22) has a negative highest order (the third order) term. Therefore, to prove that equation $F(w_2) = 0$ has only one root $w_2 = \alpha$ in the range of $c_s < \alpha < w_1$, it is sufficient to show that $F(c_s) > 0$ and $F(w_1) < 0$.

The fact that $F(c_s) > 0$ is observed from (22) by substituting w_2 with c_s. This reduces (22) to $F(c_s) = d^*(w_1 - c_s)[t_1 - t_4]'_{w_2} + U(t_f - t_s) - U(t_3 - t_2) + d^{**}(t_3 - t_2)$, which is positive if $[-t_4 + t_1]'_{w_2} > 0$.

Calculating the derivative of t_1 with respect to w_2 we obtain

$$[t_1]'_{w_2} = \left[h^-(b_2 - b_1)\sqrt{D_1^*}\right]^{-1}\left(U - \frac{b_2(w_1 - w_2)}{2} - (a_2 - b_2(c_r + w_1))\right)(b_2 - b_1),$$

which is always positive as $U \geqslant a_2$ and $\frac{b_2(w_1-w_2)}{2} < b_2(c_r + w_1)$. Calculating the derivative of t_4 with respect to w_2 we find

$$[-t_4]'_{w_2} = \left[h^+(b_2 - b_1)\sqrt{D_2^*}\right]^{-1}\left(U - \frac{1}{2}a_2 + \frac{b_2(w_1 - w_2)}{2} + \frac{1}{2}b_2(c_r + w_1)\right)(b_2 - b_1),$$

which is always positive as well. Thus, we conclude $F(c_s) > 0$.

Similarly, from (22), we find

$$\begin{aligned}F(w_2) &= d^*(w_1 - c_s)[t_1 - t_4]'_{w_2} + U(t_f - t_s) + U(w_2 - c_s)[(t_2 - t_3)]'_{w_2} - U(t_3 - t_2)\\ &\quad + d^{**}(w_2 - c_s)[(t_3 - t_2)]'_{w_2} - \frac{b_2}{2}(t_3 - t_2)(w_2 - c_s) + (t_3 - t_2)d^{**}.\end{aligned}$$

Since $[t_2]'_{w_2} = [t_1]'_{w_2} - \frac{1}{h^-}$ and $[t_3]'_{w_2} = [t_4]'_{w_2} + \frac{1}{h^+}$, we have

$$\begin{aligned}F(w_2) &= d^*(w_1 - c_s)[t_1 - t_4]'_{w_2} + U(t_f - t_s) + U(w_2 - c_s)\left([(t_1 - t_4)]'_{w_2} - \left[\frac{1}{h^-} + \frac{1}{h^+}\right]\right) - U(t_3 - t_2)\\ &\quad - d^{**}(w_2 - c_s)\left([(t_1 - t_4)]'_{w_2} - \left[\frac{1}{h^-} + \frac{1}{h^+}\right]\right) - \frac{b_2}{2}(t_3 - t_2)(w_2 - c_s) + (t_3 - t_2)d^{**}.\end{aligned}$$

Then substituting w_2 with w_1 and denoting

$$\begin{aligned}UR_2 &= U(t_3 - t_2) - d^*(w_1 - c_s)[t_1 - t_4]'_{w_2} - (U - \hat{d})(w_1 - c_s)\left([(t_1 - t_4)]'_{w_2} - \left[\frac{1}{h^-} + \frac{1}{h^+}\right]\right)\\ &\quad - \left(\hat{d} - \frac{b_2}{2}(w_1 - c_s)\right)(t_3 - t_2),\end{aligned}$$

where $\hat{d} = \frac{a_2 - b_2(c_r + w_1)}{2}$ and requiring $F(w_1) < 0$, we have $t_f - t_s \leqslant R_2$ as stated in the lemma. Finally, recalling that according to Lemma 3, $t_3 \geqslant t_2$, we find

$$t_3 - t_2 = t_f - t_s + A_1^* + A_2^* - (w_1 - w_2)\left(\frac{1}{h^-} + \frac{1}{h^+}\right) \geqslant 0.$$

Thus, denoting, $R_1 = -A_1^* - A_2^* + (w_1 - w_2)(\frac{1}{h^-} + \frac{1}{h^+})$, we require $t_f - t_s \geqslant R_1$. □

Proof of Theorem 3. The proof is immediate. According to Lemma 5, $F(w_2) = 0$ has only one root in the feasible range of $c_s < \alpha < w_1$, therefore the optimal wholesale price it defines is unique. Furthermore, according to Lemmas 3 and 4, $p^*(t)$ and $u^*(t)$ are unique and feasible if $t_3 \geqslant t_2$ and $a(t) - b(t)(c_r + w(t)) \geqslant 0$ hold. Substituting into the latter the corresponding values for $b(t)$ and $w(t)$, we obtain the conditions stated in Theorem 3. □

Proof of Lemma 6. It is shown in Lemma 3 that the demand increases starting from t_1 and returns to the steady-state level at t_4, so that $\int_{t_1}^{t_4} u(t)\,dt = \int_{t_1}^{t_4} d(t)\,dt$. Since $u(t) = 0$ for $t_1 \leqslant t < t_s$ and $t_f \leqslant t < t_4$, i.e. $\int_{t_1}^{t_4} u(t)\,dt = \int_{t_s}^{t_f} u(t)\,dt$ and the demand in the transient state during these intervals of time is greater than the demand in the steady-state, it is sufficient to estimate an increase in the total order quantity during period $[t_s, t_f]$. Specifically, substituting the minimal demand at $t = t_s$ and $t = t_f$ (see Lemma 3), $a_2 - b_2 p$, where $p = p(t_s) = p(t_f)$, we find for the transient state $\int_{t_1}^{t_4} u(t)\,dt > \int_{t_s}^{t_f} d(t)\,dt > (a_2 - b_2 p)(t_f - t_s)$, $p = \frac{a_2 + b_2(c_r + w_1 - h^-(t_s - t_1))}{2b_2} < \frac{a_2 + b_2(c_r + w_1)}{2b_2}$. The total order in a steady-state is $\int_{t_s}^{t_f} d(t)\,dt = (a_1 - b_1 p^*)(t_f - t_s)$, $p^* = \frac{a_1 + b_1(c_r + w_1)}{2b_1}$(see Lemma 1). Thus, the difference per time unit is $(a_2 - b_2 p) - (a_1 - b_1 p^*) \geqslant \frac{1}{2}(a_2 - b_2(c_r + w_1) - (a_1 - b_1(c_r + w_1)))$. Next, taking into account that $a_1 = g + b_1 P$ and $a_2 = g + b_2 P$, we have per time unit, $(a_2 - b_2 p) - (a_1 - b_1 p^*) \geqslant \frac{1}{2}((P - c_r - w_1)(b_2 - b_1)) > 0$, as stated in this lemma, where $w^*(t) = w_1 = \frac{a_1 - b_1(c_r - c_s)}{2b_1}$ (see Theorem 1). □

Proof of Lemma 7. The proof is similar to that of Lemma 3. The difference is that the solution defined by Lemma 3 will remain optimal under random disturbances with respect to the stochastic conditions (16) and (17) if the breaking points do not change, that is

$$X(t_2) = -\int_{t_1}^{t_2} (a(t) + e_t(\xi) - b(t)p(t,\xi))\,dt + U(t_2 - t_s) = 0,$$

$$X(t_4) = U(t_f - t_3) - \int_{t_3}^{t_4} (a(t) + e_t(\xi) - b(t)p(t,\xi))\,dt = 0 \tag{A.14}$$

and $E_{R(t,\xi)}[\psi(t,\xi)] = \psi(t)$, where $\psi(t)$ is determined by (A.10) and (A.11). The latter is ensured for a steady-state by the conditions derived in Lemma 2 and for a transient state by $X(t,\xi) < 0$ for $t_1 < t < t_2$, $X(t,\xi) > 0$ for $t_3 < t < t_4$ (see conditions (10)). Thus, we need that in addition to (A.14) (which with respect to (A.12) and (A.13) simplifies to $\int_{t_1}^{t_2} e_t(\xi)\,dt = \int_{t_3}^{t_4} e_t(\xi)\,dt = 0$), the following holds

$$X(t,\xi) = -\int_{t_1}^{t} (a(\tau) + e_\tau(\xi) - b(\tau)p(\tau,\xi))\,d\tau + U(t - t_s) \leqslant 0 \quad \text{for } t_1 < t < t_2 \quad \text{and}$$

$$X(t) = U(t - t_3) - \int_{t_3}^{t} (a(\tau) + e_\tau(\xi) - b(\tau)p(\tau,\xi))\,dt \geqslant 0 \quad \text{for } t_3 < t < t_4,$$

as stated in the lemma. □

Proof of Theorem 4. Similar to Theorem 2, we note that calculating the objective function (2) and taking into account Lemma 7 and $E[e_t] = 0$, we obtain (21) and again the optimality condition (22). Therefore, the proof can be concluded with the same arguments as for Theorem 3. □

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