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Optimal Finite-Horizon Production Control in a Defect-Prone Environment

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***Abstract*—In this note, we consider a single-machine, single-part-type production system, operating in a defect-prone environment. It is assumed that there is a random yield proportion of nondefective parts, with known probability distribution. Over each production cycle, it is assumed that there is a single realization of the yield random variable. Furthermore, it is assumed that the system is operated under a periodic-review policy. Thus, the particular realization of the yield proportion cannot be determined prior to the end of the production horizon. The optimal production control, that minimizes a linear combination of expected surplus and shortage costs over the planning horizon is shown to be piecewise constant, and the appropriate production levels and control break-points are determined as functions of the yield rate distribution.**

***Index Terms*—Cost minimization, defect-prone, finite-horizon, production control, random yield.**

I. Introduction

In this note, we provide analytical results for a basic manufacturing system model where the effect of the yield uncertainty is introduced. It is assumed that the probability distribution of the random yield rate is known, but the inventory level is observable only intermittently. The optimal production control, that minimizes a linear combination of expected surplus and shortage costs over the planning horizon, is shown to be piecewise constant, and the appropriate production levels and control break-points are determined as functions of the yield rate distribution. It is interesting to note that, even for the one-machine, one-part-type system, the consideration of random yield leads to a nonintuitive, and nontrivial, optimal production control.

The incorporation of random yield into manufacturing system models has been of interest since as early as 1958 (see [9]). Since then, many authors have considered random yield problems in various forms. In 1995, Yano and Lee [13] provided a comprehensive review of the existing literature. Based on the system modeling characteristics, they arranged random yield lot-sizing problems into the following categories: discrete-time models, which include single-stage models (both single and multiple period), multiple stages in tandem, assembly systems, and continuous-time models with constant demand rates or random demand rates.

Subsequently, more generic models have been studied including extensions such as uncertain supply, backlogged demand, imperfect production, and late-stage inspection. Yield variability due to random production capacity is considered in [2] and in [8]. More recently, Grosfeld-Nir and Gerchak [4] examined a model with multiple successive production runs to meet orders. They focused on relatively small volumes of custom-made items and analyzed the yield structure. Liu and Yang [11] considered multiple defect types (reworkable and nonreworkable defects) and determined the optimal lot size. Bollapragada and Morton [1] used myopic heuristics for the random yield problem and obtained promising results. Yao and Zheng [14] studied a two-stage problem in which, in order to coordinate the inspection procedures at the two stages, the optimal policy is characterized by a sequence of thresholds at stage 1 and by a priority structure at stage 2. For an assembly system under random demand and production yield loss, Gurnani *et al.* [5] circumvented the difficulty of solving the original problem by modifying the exact cost function with an approximate one and determined a bound on the difference. Grosfeld-Nir *et al.* [3] included inspection costs as a key part of the problem in a general multiple production run model.

A related problem is the extension of the classical single-period newsboy problem to incorporate yield variability. (See [10] for an excellent survey of the newsboy problem and its extensions.) For example, Henig and Gerchak [7] examine the single-period newsboy problem with random yield. Sipper and Bulfin, Jr. [12] discuss the implications of random yield on material resource planning.

A primary difference between the model considered in this note and the newsboy problem is that the newsboy problem assigns costs, and allows replenishment, at discrete points of time. However, our model considers continuous production and assignment of costs, although the exact inventory position is known only periodically. In effect, the traditional random yield lot-sizing problem is transformed into a continuous-time optimal control problem.

A constructive approach will be used to analyze the problem with the aid of the maximum principle. The maximum principle is a set of optimality conditions, the application of which results in a new objective function, called the Hamiltonian, and a co-state differential equation (see, for example, [6]). A solution (control function) will be constructed that is both feasible and maximizes the Hamiltonian. This ensures optimality.

The remainder of the note is organized as follows. In Section II, the single part-type problem is described, and the original stochastic problem is transformed into an equivalent deterministic problem. Section III details the dual formulation, deriving the co-state equation and describing the form of the optimal control. Then, in Section IV, the optimal control is determined analytically for the piecewise linear instantaneous cost function.

II. Single-Part-Type Problem Description

Consider a single-machine, single-part-type production system, operating in a defect-prone environment. Suppose the production rate of the system, $u(t)$, is bounded and controllable, i.e.,

$$0 \le u(t) \le U. \tag{1}$$

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Also, assume that the material flow is approximated by a fluid, and the demand rate Δ is constant over some fixed production horizon $[0, T]$. Then, the inventory process, $X(t)$, may be described by the following dynamics:

$$\dot{X}(t) = \boldsymbol{\alpha} u(t) - \Delta \quad X(0) = x_0 \tag{2}$$

where $\boldsymbol{\alpha}$ is a random variable representing the yield proportion of nondefective parts and is characterized by the continuous probability density function $f(\alpha)$. For each production horizon, there will be a single realization of this yield random variable. Furthermore, assume that the detection of any defects requires at least time T. Thus, the particular realization of $\boldsymbol{\alpha}$ which will hold on $[0, T]$ cannot be determined prior to the end of the production horizon. Since no new information other than $f(\alpha)$ will become available during the production horizon, the determination of how much to produce and when to produce it must be made under these uncertain conditions, before production commences.

The objective is to determine $\{u(t) : 0 \leq t \leq T\}$, the rate of production over the entire production horizon, in order to minimize the resulting total expected cost

$$J(u) = E\left[\int_0^T g\left(X(t)\right) dt\right] \to \min \tag{3}$$

where $g(\cdot)$ is the instantaneous buffer-level inventory cost function. It is assumed that $g(\cdot)$ is a nonnegative, continuous, piecewise differentiable, strictly convex function, with $g(0) = 0$.

In order to derive an equivalent deterministic problem, we introduce a new state variable, $Y(t)$, which represents the cumulative production at time t, i.e.,

$$\dot{Y}(t) = u(t), \text{ with } Y(0) = 0. \tag{4}$$

Integrating (2), and substituting the integral of (4) into the result, yields

$$X(t) = x_0 - t\Delta + \boldsymbol{\alpha} \int_0^t u(\tau) d\tau = x_0 - t\Delta + \boldsymbol{\alpha} Y(t). \tag{5}$$

Substituting (5) into (3) shows that (1)–(3) are equivalent to

$$J(u) = \int_0^T \left\{ \int_0^1 g\left(x_0 - t\Delta + \alpha Y(t)\right) f(\alpha) d\alpha \right\} dt \to \min \tag{6}$$

subject to (1) and (4).

III. Dual Formulation

Given (1), (4), and (6), we construct the Hamiltonian as follows:

$$H(u, t) = -\int_0^1 g\left(x_0 - t\Delta + \alpha Y(t)\right) f(\alpha) d\alpha + \psi(t) u(t) \tag{7}$$

where $\psi(t)$ is the co-state variable. According to the maximum principle (see, for example, [6]), the co-state variable is absolutely continuous and satisfies the co-state (dual) equation with transversality (boundary) condition

$$\begin{aligned} \dot{\psi}(t) &= -\frac{\partial H(u,t)}{\partial Y(t)} \\ &= \int_0^1 \alpha f(\alpha) \left.\frac{\partial g(s)}{\partial s}\right|_{s = x_0 - t\Delta + \alpha Y(t)} d\alpha \\ &\quad \text{where } \psi(T) = 0. \end{aligned} \tag{8}$$

The Hamiltonian is the objective of the dual problem, and it should be maximized at each point of time with respect to the admissible control set. Since only the second term of (7) explicitly depends on $u(t)$, the maximization implies that

$$u(t) = \begin{cases} U, & \text{if } \psi(t) > 0 \quad \text{(i)} \\ 0, & \text{if } \psi(t) < 0 \quad \text{(ii)} \\ u \in [0, U], & \text{if } \psi(t) = 0 \quad \text{(iii)} \end{cases}. \tag{9}$$

Given the function $\psi(t)$, the co-state variable-based necessary optimality conditions (9)(i) and (ii) describe the full-production and no-production regions, respectively; that is, when the production rate reaches its maximum and its minimum (in our case 0). However, the optimal control is not necessarily "bang–bang" (we will see that, in general, it is not), due to condition (9)(iii), which describes the singular production region. In order to resolve the singular production ambiguity of the optimal control, it is necessary to have additional information on the shape of the instantaneous cost function $g(\cdot)$. Therefore, in the remainder of this note, we will consider the linear, absolute value cost function.

IV. Piecewise Linear Instantaneous Cost

A. *Properties of the Primal and Dual Formulations*

Consider the piecewise linear instantaneous cost function

$$g\left(X(t)\right) = c^+ X^+(t) + c^- X^-(t) \tag{10}$$

where c^+ and c^- are the positive inventory surplus and shortage cost coefficients, respectively, $X^+(t) = \max\{0, X(t)\}$ and $X^-(t) = \max\{0, -X(t)\}$.

Primal Formulation: Given $g(X(t)) = c^+ X^+(t) + c^- X^-(t)$, the objective function (6) may be written as

$$\begin{aligned} J(u) = \int_0^T \Bigg[& \int_{\frac{t\Delta - x_0}{Y(t)}}^{\infty} c^+ \left(x_0 - t\Delta + \alpha Y(t)\right) f(\alpha) d\alpha \\ & - \int_{-\infty}^{\frac{t\Delta - x_0}{Y(t)}} c^- \left(x_0 - t\Delta + \alpha Y(t)\right) f(\alpha) d\alpha \Bigg] dt. \end{aligned} \tag{11}$$

Lemma 4.1: Together, (1), (4), and (11) constitute a convex program. Thus, there is only one optimal value for the objective function.

Proof: Denote the integrand in (11) as $R(t)$, so that $J(u) = \int_0^T R(t) dt$. Since constraints (1) and (4) are linear, the proof is completed by verifying that the objective (11) is convex with respect to $Y(t)$, i.e.,

$$\frac{\partial^2 R(t)}{\partial Y(t)^2} = (c^+ + c^-) f\left(\frac{t\Delta - x_0}{Y(t)}\right) \frac{(t\Delta - x_0)^2}{Y^3(t)} \geq 0.$$

□

Dual Formulation: For the piecewise linear instantaneous cost function (10), the co-state (8) becomes

$$\dot{\psi}(t) = \int_{Z(t)}^{\infty} c^+ \alpha f(\alpha) d\alpha - \int_{-\infty}^{Z(t)} c^- \alpha f(\alpha) d\alpha, \text{ and } \psi(T) = 0 \tag{12}$$

where $Z(t) = (t\Delta - x_0)/Y(t)$. Throughout the remainder of this note, we will use this normalized variable $Z(t)$, which will be referred to as the *state index* because it involves the state variable $Y(t)$. As is apparent from (12), the state index $Z(t)$ will play a critical role in the subsequent analysis. If $Z(t) < 0$, then there has been an inventory surplus up until time t, i.e., $X(s) > 0$ for $0 \leq s \leq t$. In this case, the second interior integral in the objective function (11) vanishes for $0 \leq s \leq t$. If $Z(t) > 1$, then there is an inventory shortage at time t, i.e., $X(t) < 0$. However, we cannot immediately deduce that there has been an inventory shortage up until time t.

Theorem 4.1: The optimality conditions (4), (9), and (12) are both necessary and sufficient.

Proof: The conclusion is immediate from Lemma 4.1. □

Given (12), we now can resolve the ambiguity of the singular production condition (9)(iii).

Lemma 4.2: Consider some t_1, t_2 such that $0 \le t_1 < t_2 \le T$. Suppose that $\psi(t) = 0$ for $t \in \tau = (t_1, t_2)$. Then, there exists some constant $\beta \in (0, 1)$, such that, for $t \in \tau$, $Z(t) = \beta$ and $u(t) = \Delta/\beta$, where

$$\int_0^{\beta} \alpha f(\alpha) d\alpha = \frac{c^+}{c^+ + c^-} E[\boldsymbol{\alpha}]. \tag{13}$$

Proof: Differentiating $\psi(t) = 0$ on the interval τ, and substituting (12) into the result yields

$$\int_{Z(t)}^{\infty} c^+ \alpha f(\alpha) d\alpha - \int_{-\infty}^{Z(t)} c^- \alpha f(\alpha) d\alpha = 0. \tag{14}$$

Although (14) cannot be solved explicitly for an arbitrary function $f(\cdot)$, it can be simplified. By dividing the first term of (14) into $\int_{-\infty}^{\infty} c^+ \alpha f(\alpha) d\alpha - \int_{-\infty}^{Z(t)} c^+ \alpha f(\alpha) d\alpha$ and recombining the terms, using the fact that $0 \le \alpha \le 1$, yields

$$\int_0^{Z(t)} \alpha f(\alpha) d\alpha = \frac{c^+}{c^+ + c^-} E[\boldsymbol{\alpha}]. \tag{15}$$

The right-hand side of (15) is a constant. Thus, $Z(t)$ must be a constant, say β. In addition, since $\int_0^1 \alpha f(\alpha) d\alpha = E[\boldsymbol{\alpha}] > 0$, it follows that $0 < \beta < 1$. Furthermore, differentiating $Z(t) = (t\Delta - x_0)/Y(t) = \beta$ yields $u(t) = \Delta/\beta$ for $t \in \tau$. □

Example 4.1: Consider a reverse, truncated exponential distribution on the interval [0,1], i.e.,

$$f(\alpha) = \begin{cases} \frac{\lambda e^{\lambda\alpha}}{e^{\lambda} - 1}, & \text{for } 0 \le \alpha \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

For this yield probability density function, we have

$$\int_0^{\beta} \alpha f(\alpha) d\alpha = \frac{1}{\lambda(e^{\lambda} - 1)} \left[1 + (\beta\lambda - 1)e^{\beta\lambda}\right].$$

Note that $E[\boldsymbol{\alpha}]$ can then be obtained by setting $\beta = 1$. Thus, (13) becomes

$$\frac{1}{\lambda(e^{\lambda} - 1)} \left[1 + (\beta\lambda - 1)e^{\beta\lambda}\right] = \frac{c^+}{c^+ + c^-} \frac{1}{\lambda(e^{\lambda} - 1)} \left[1 + (\lambda - 1)e^{\lambda}\right].$$

Simplifying this, we obtain the transcendental equation

$$(1 - \beta\lambda)e^{\lambda\beta} = \frac{c^- + (1 - \lambda)e^{\lambda} c^+}{c^+ + c^-}.$$

Although the aforementioned equation cannot be solved analytically for β, it may easily be solved numerically to any desired precision.

Example 4.2: Suppose α is uniformly distributed on the unit interval [0,1]. Then, since $E[\boldsymbol{\alpha}] = 1/2$, (13) becomes

$$\frac{c^+}{2(c^+ + c^-)} = \int_0^{\beta} \alpha d\alpha = \frac{1}{2}\beta^2.$$

Solving for β yields $\beta = \sqrt{c^+/(c^+ + c^-)}$.

This subsection provides the theoretic preparation for determining the optimal control trajectory $u^*(t)$ for the production system given by (1) and (2). However, to complete the solution, it is necessary to consider two cases, defined as follows: If a manufacturing system has sufficient available capacity to produce at rate Δ/β, then the system will be referred to as **nondeficient**; if it does not have adequate capacity to produce at rate Δ/β, then it will be referred to as **deficient**.

In the following section, we solve the problem for a nondeficient system first. Then, based on the results in Section IV-B, we complete the solution for a deficient system in Section IV-C.

B. Nondeficient Systems

1) Optimal Control for Nondeficient Systems: Suppose that there is enough capacity to produce at the level suggested by Lemma 4.2, i.e., $\Delta/\beta \le U$. We consider two cases, depending on whether there is an initial inventory surplus or shortage. The optimal solution for these cases is proved using a constructive two-step approach. First, we propose a solution, which satisfies the necessary and sufficient optimality conditions (9)(i)–(iii). Then, we verify whether this candidate solution is feasible. If the proposed solution is feasible, then it is globally optimal.

*Lemma 4.3: (**Initial Inventory Surplus**):* Suppose $x_0 \ge 0$ and $\Delta/\beta \le U$. If $x_0/\Delta \ge T$, then it is optimal not to produce any product, i.e., $u(t) = 0$ for $0 \le t \le T$. If $x_0/\Delta < T$, then the optimal production control is given by

$$u(t) = \begin{cases} 0, & \text{for } 0 \le t < \frac{x_0}{\Delta} \\ \frac{\Delta}{\beta}, & \text{for } \frac{x_0}{\Delta} \le t \le T \end{cases}.$$

Proof: The proof is contained in the Appendix. □

*Lemma 4.4: (**Initial inventory shortage**):* Suppose $x_0 < 0$ and $\Delta/\beta \le U$. If $-x_0/(\beta U - \Delta) \ge T$, then it is optimal to produce at the maximum rate over the entire horizon, i.e., $u(t) = U$ for $0 \le t \le T$. If $-x_0/(\beta U - \Delta) < T$, then the optimal production control is given by

$$u(t) = \begin{cases} U, & \text{for } 0 \le t < \frac{-x_0}{\beta U - \Delta} \\ \frac{\Delta}{\beta}, & \text{for } \frac{-x_0}{\beta U - \Delta} \le t \le T \end{cases}.$$

Proof: The proof is contained in the Appendix. □

C. Deficient Systems

As discussed earlier, for systems that are not deficient, the feasibility of a candidate solution, satisfying the optimality conditions (9)(i)–(iii), can be verified explicitly. However, for deficient systems, the determination of a switching point, i.e., the point at which the piecewise constant control changes from one value to another, requires solving a nonlinear equation similar to (12). Thus, for deficient systems, it is not possible to verify the feasibility of a candidate solution explicitly using (9)(i)–(iii). However, as will be shown, it is possible to verify feasibility by excluding all infeasible trajectories. Then, we will show that the only remaining solution satisfies (9)(i)–(iii) and is therefore necessarily feasible because the original problem always has a solution.

1) Optimal Control for Deficient Systems: We begin by proving the following general property, which holds for all systems, whether or not they are deficient.

Lemma 4.5: If, under an optimal production control, the machine is idle over any measurable interval of time $(t_1, t_2]$, where $0 \le t_1 < t_2 \le T$, then the machine must have been idle since the beginning of the production horizon, i.e., $u(t) = 0$ for $0 \le t \le t_2$.

Proof: The proof is by contradiction. Suppose that the production rate is initially either U or Δ/β, and the system becomes idle after some time $t_1 > 0$. In addition, assume the co-state variable satisfies (12). Then, according to (9), $\psi(t) \ge 0$ prior to time t_1, and then $\psi(t) < 0$ after t_1.

Differentiating (8) yields

$$\ddot{\psi}(t) = (c^+ + c^-)\left(\frac{t\Delta - x_0}{Y^2(t)}u(t) - \frac{\Delta}{Y(t)}\right) \times f\left(\frac{t\Delta - x_0}{Y(t)}\right)\frac{t\Delta - x_0}{Y(t)}.$$

Since, from the beginning of the production horizon, $u(t)$ is equal to either U or Δ/β, it follows that $Y(t) = tu(t) > 0$ for $t > 0$. Thus, $\ddot{\psi}(t)$ is a measurable, bounded function, and $\dot{\psi}(t)$ is a continuous function.

Since $\dot{\psi}(t)$ is continuous, and $\psi(t)$ changes from nonnegative to negative at t_1, this implies $\dot{\psi}(t) < 0$ starting from some point $t_0 \le t_1$. Thus, using (12), it follows that

$$\int_{Z(t)}^{1} c^+\alpha f(\alpha)d\alpha - \int_{0}^{Z(t)} c^-\alpha f(\alpha)d\alpha < 0, \text{ for } t_0 \le t \le t_1.$$

This is possible only if $Z(t) > 0$ for $t_0 \le t \le t_1$. However, the system is idle for $t > t_1$, which implies $Y(t)$ remains constant, i.e., $Z(t) = (t\Delta - x_0)/Y(t) > 0$ increases and, hence, $\psi(t) < 0$ decreases. Thus, the transversality condition $\psi(T) = 0$ from (12) will never be met. □

Corollary 4.1: Under an optimal production control, once the machine commences production, it will continue to produce material throughout the remainder of the production horizon.

Proof: The proof is an immediate consequence of Lemma 4.5.□

The following two lemmas study two possible cases of deficient systems $(\Delta/\beta) > U$ characterized by an initial inventory surplus, $x_0 \ge 0$. The first case is further referred to as completely deficient, $\Delta > U$, while the other as fairly deficient, $\Delta \le U < \Delta/\beta$.

The optimality of the no-production control for a sufficiently large initial inventory surplus, i.e., if $x_0/\Delta \ge T$, was proven in Lemma 4.3 without assuming any relationship between U and Δ. Therefore, this result continues to hold for completely deficient and fairly deficient systems. Next, we consider the case of a completely deficient system with $x_0/\Delta < T$.

Lemma 4.6: Consider a system with $0 \le x_0/\Delta < T$ and $\Delta > U$. Let

$$t_2 = \frac{c^+E[\boldsymbol{\alpha}]\frac{x_0}{U} + c^-E[\boldsymbol{\alpha}]T - (c^+ + c^-)B\left(\alpha, \frac{\Delta}{U}\right)\frac{x_0}{U}}{c^+E[\boldsymbol{\alpha}]\frac{\Delta}{U} + c^-E[\boldsymbol{\alpha}] - (c^+ + c^-)B\left(\alpha, \frac{\Delta}{U}\right)\frac{\Delta}{U}} \tag{16}$$

where

$$B\left(\alpha, \frac{\Delta}{U}\right) = \int_0^1 \frac{1-\alpha}{\frac{\Delta}{U} - \alpha}\alpha f(\alpha)d\alpha.$$

If $t_2 < T$, solve for t_1 from

$$\frac{t_2\Delta - x_0}{U(t_2 - t_1)} = 1. \tag{17}$$

Otherwise, for $t_2 \ge T$, solve for t_1 from

$$c^+E[\boldsymbol{\alpha}](T - t_1) - (c^+ + c^-)\int_{\frac{x_0}{\Delta}}^{T}\int_{0}^{\frac{t\Delta - x_0}{U(t-t_1)}} \alpha f(\alpha)d\alpha dt = 0. \tag{18}$$

For either case (17) or (18), let $\hat{t}_1 = \max(t_1, 0)$. Then, the optimal production control is

$$u(t) = \begin{cases} 0, & \text{for } 0 \le t < \hat{t}_1 \\ U, & \text{for } \hat{t}_1 \le t \le T \end{cases}.$$

Proof: The proof is contained in the Appendix. □

Lemma 4.7: Consider a system with $0 \le x_0/\Delta < T$ and $\beta U < \Delta \le U$. Define time t_1 to satisfy

$$c^+E[\boldsymbol{\alpha}](T - t_1) - (c^+ + c^-)\int_{\frac{x_0}{\Delta}}^{T}\int_{0}^{\frac{t\Delta - x_0}{U(t-t_1)}} \alpha f(\alpha)d\alpha dt = 0.$$

If $t_1 > 0$, then the optimal production control is given by

$$u(t) = \begin{cases} 0, & \text{for } 0 \le t \le t_1 \\ U, & \text{for } t_1 < t \le T. \end{cases}$$

Otherwise, for $t_1 \le 0$, the full-production control is optimal, i.e., $u(t) = U$ for $0 \le t \le T$.

Proof: If $t_1 \le 0$, the argument is same as was used as in the previous lemma. Thus, suppose $t_1 > 0$. Under the proposed control $u(t)$, $Z(t)$ is nondecreasing. In addition

$$\begin{aligned} &Z(t) = -\infty, && \text{for } 0 \le t \le t_1 \\ &-\infty < Z(t) \le 0, && \text{for } t_1 < t \le \tfrac{x_0}{\Delta}. \\ &0 < Z(t) \le 1 && \text{for } \tfrac{x_0}{\Delta} < t \le T \end{aligned} \tag{9}$$

Note that $Z(T) \le 1$, since $(x_0 - t_1\Delta) + (U - \Delta)(T - t_1) \ge 0$. This is the same as the second subcase of the general sequence, which was considered in the previous lemma. Thus, the optimal control will be the same as was obtained there. □

Next, consider a deficient system having an initial backlog. The optimal production control in this case does not depend on whether the system is fairly or completely deficient.

Lemma 4.8: Suppose $x_0 < 0$ and $\Delta/\beta > U$. Then, the optimal control is $u(t) = U$ for $0 \le t \le T$.

Proof: Since $\Delta/\beta > U$, the singular production control described by (9)(iii) is infeasible. According to Corollary 4.1, machine idleness cannot follow production at full rate. We will show that, for $x_0 < 0$ and $\Delta/\beta > U$, an optimal trajectory will never begin with no-production. Therefore, there is only one possible regime left for an optimal trajectory, which is production at full rate.

The remainder of the proof is by contradiction. We assume that the system is idle for $0 \le t \le t_1$ and the co-state variable has a feasible behavior satisfying (12). Then, $Y(t) = 0$, $Z(t) = (t\Delta - x_0)/Y(t) = \infty$, $0 \le t \le t_1$ and according to (9), $\psi(t) < 0$, $0 \le t \le t_1$. Consequently, (12) becomes

$$\dot{\psi}(t) = -\int_{-\infty}^{Z(t)} c^-\alpha f(\alpha)d\alpha = -\int_0^1 c^-\alpha f(\alpha)d\alpha = -c^-E[\boldsymbol{\alpha}].$$

Thus, $\psi(t) < 0$ and $\dot{\psi}(t) < 0$, which according to (9) means no production, i.e., $Y(t) = 0$ and $\psi(t)$ will only decrease until the end of the production horizon. Thus, the transversality condition from (12) will never be satisfied. □

Appendix I

Proof of Lemma 4.3

The case in which $x_0/\Delta \ge T$ is trivial, since, regardless of the control implemented, there is no possibility of an inventory shortage. Thus, it is optimal not to produce any product.

Next, suppose that $x_0/\Delta < T$. The candidate solution is

$$u(t) = \begin{cases} 0, & \text{for } 0 \le t < \frac{x_0}{\Delta} \\ \frac{\Delta}{\beta}, & \text{for } \frac{x_0}{\Delta} \le t \le T \end{cases}.$$

Since $Y(t) = \int_0^t u(\tau)d\tau$, this implies that

$$Y(t) = \begin{cases} 0, & \text{for} 0 \le t < \frac{x_0}{\Delta} \\ \frac{\Delta}{\beta}\left(t - \frac{x_0}{\Delta}\right), & \text{for} \frac{x_0}{\Delta} \le t \le T \end{cases}.$$

Substituting this proposed solution into the state index $Z(t) = (t\Delta - x_0)/Y(t)$, and using L'Hôpital's rule at $t = x_0/\Delta$, it follows that

$$\begin{array}{ll} Z(t) = -\infty, & \text{for} \quad 0 \le t < \frac{x_0}{\Delta} \\ 0 < Z(t) = \beta < 1, & \text{for} \quad \frac{x_0}{\Delta} \le t \le T \end{array}.$$

Consequently, by integrating the co-state (12) and using Lemma 4.2, we obtain

$$\psi(t) = \begin{cases} \psi(0) + \int_0^t \int_{Z(t)}^{\infty} c^+ \alpha f(\alpha) d\alpha d\tau & \\ \quad = \psi(0) + c^+ E[\boldsymbol{\alpha}]t, & \text{for } 0 \le t < \frac{x_0}{\Delta} \\ 0, & \text{for } \frac{x_0}{\Delta} \le t \le T \end{cases}.$$

Choosing $\psi(0) = -(c^+ x_0/\Delta)E[\boldsymbol{\alpha}]$ yields

$$\begin{aligned} \psi(t) &= -\frac{c^+ x_0}{\Delta} E[\boldsymbol{\alpha}] + c^+ E[\boldsymbol{\alpha}]t \\ &= c^+ E[\boldsymbol{\alpha}] \left(t - \frac{x_0}{\Delta}\right) < 0, \text{ for } 0 \le t < \frac{x_0}{\Delta}. \end{aligned}$$

This solution is feasible, and it also satisfies the optimality condition (9) over the entire production horizon. Thus, according to Lemma 4.1 and Theorem 4.1, it is globally optimal. □

Proof of Lemma 4.4

Suppose $-x_0/(\beta U - \Delta) \ge T$ first. Since $0 < \beta < 1$, we can separate this into two sub-cases: $0 \le T \le -x_0/(U - \Delta)$ and $-x_0/(U - \Delta) < T \le -x_0/(\beta U - \Delta)$. For the case that $0 \le T \le -x_0/(U - \Delta)$, the production horizon is not long enough to erase the inventory shortage, regardless of the chosen rate of production. Thus, the full-production control is optimal.

Consider the second subcase where $-x_0/(U - \Delta) < T \le -x_0/(\beta U - \Delta)$. As in the previous subcase, $Z(t) \ge 1$ on the interval $0 \le t \le -x_0/(U - \Delta)$. Thus, from (12), it follows that

$$\psi(t) = \psi(0) - \int_0^t \int_{-\infty}^{Z(t)} c^- \alpha f(\alpha) d\alpha d\tau = \psi(0) - c^- E[\boldsymbol{\alpha}]t \tag{20}$$

on this interval. On the interval $-x_0/(U - \Delta) \le t \le -x_0/(\beta U - \Delta)$, also using $u(t) = U$, it follows that $Z(t) = \Delta/U - x_0/Ut$, which is decreasing in t. Note that at times $t = -x_0/(U - \Delta)$ and $t = -x_0/(\beta U - \Delta)$, $Z(-x_0/(U - \Delta)) = 1$ and $Z(-x_0/(\beta U - \Delta)) = \beta$. It follows that $\beta \le Z(t) \le 1$ on the interval $-x_0/(U - \Delta) \le t \le T$. Therefore, on this interval, (12) may be rewritten as

$$\begin{aligned} \dot{\psi}(t) = \int_0^1 c^+ \alpha f(\alpha) d\alpha - \int_0^{\beta} c^+ \alpha f(\alpha) d\alpha - \int_{\beta}^{Z(t)} c^+ \alpha f(\alpha) d\alpha \\ - \int_0^{\beta} c^- \alpha f(\alpha) d\alpha - \int_{\beta}^{Z(t)} c^- \alpha f(\alpha) d\alpha. \end{aligned}$$

Using (13)

$$\dot{\psi}(t) = -(c^+ + c^-) \int_{\beta}^{Z(t)} \alpha f(\alpha) d\alpha.$$

Integrating this equation, and using the terminal condition $\psi(T) = 0$, yields

$$\psi(t) = (c^+ + c^-) \int_t^T \int_{\beta}^{Z(s)} \alpha f(\alpha) d\alpha ds \ge 0, \text{ for } \frac{-x_0}{U - \Delta} \le t \le T. \tag{21}$$

From the continuity of $\psi(t)$ at the point $-x_0/(U - \Delta)$, and using (20) and (21), it follows that

$$\psi(0) = c^- E[\boldsymbol{\alpha}] \frac{-x_0}{U - \Delta} + (c^+ + c^-) \int_{\frac{-x_0}{U-\Delta}}^{T} \int_{\beta}^{Z(s)} \alpha f(\alpha) d\alpha ds.$$

Since $\psi(t)$ satisfies the transversality condition (12) and the optimality conditions (9)(i)–(iii), the proposed control is optimal for this case.

Next suppose that $-x_0/(\beta U - \Delta) < T$. Then, the candidate solution given by the lemma is

$$u(t) = \begin{cases} U, & \text{for } 0 \le t \le \frac{-x_0}{\beta U - \Delta} \\ \frac{\Delta}{\beta}, & \text{for } \frac{-x_0}{\beta U - \Delta} < t \le T \end{cases}$$

which implies

$$Y(t) = \begin{cases} Ut, & \text{for } 0 \le t \le \frac{-x_0}{\beta U - \Delta} \\ \frac{\Delta}{\beta}\left(t - \frac{x_0}{\Delta}\right), & \text{for } \frac{-x_0}{\beta U - \Delta} < t \le T \end{cases}. \tag{22}$$

Since $0 < \beta < 1$ and $\Delta/\beta \le U$, (22) implies that

$$\begin{array}{ll} Z(t) = \frac{t\Delta - x_0}{Ut} \ge \beta > 0, & \text{for} \quad 0 \le t < \frac{-x_0}{\beta U - \Delta} \\ 0 < Z(t) = \frac{t\Delta - x_0}{\frac{\Delta}{\beta}(t - \frac{x_0}{\Delta})} = \beta < 1, & \text{for} \quad \frac{-x_0}{\beta U - \Delta} \le t \le T \end{array}.$$

Thus, $\dot{\psi}(t)$ is given by

$$\dot{\psi}(t) = \begin{cases} -c^- E[\boldsymbol{\alpha}], & \text{for } 0 \le t < \frac{-x_0}{U - \Delta} \\ -(c^+ + c^-) \int_{\beta}^{Z(t)} \alpha f(\alpha) d\alpha, & \text{for } \frac{-x_0}{U - \Delta} \le t \le \frac{-x_0}{\beta U - \Delta} \\ 0, & \text{for } \frac{-x_0}{\beta U - \Delta} < t \le T \end{cases}.$$

Integrating this yields

$$\psi(t) = \begin{cases} \psi(0) - c^- E[\boldsymbol{\alpha}]t, & \text{for} 0 \le t < \frac{-x_0}{U - \Delta} \\ \psi\left(\frac{-x_0}{\beta U - \Delta}\right) + (c^+ + c^-) & \\ \quad \times \int_t^T \int_{\beta}^{Z(s)} \alpha f(\alpha) d\alpha ds, & \text{for} \frac{-x_0}{U - \Delta} \le t \le \frac{-x_0}{\beta U - \Delta} \\ 0, & \text{for} \frac{-x_0}{\beta U - \Delta} < t \le T \end{cases}.$$

Using the continuity of the co-state variable $\psi(t)$ and the terminal condition $\psi(T) = 0$, it follows that

$$\psi(0) = c^- E[\boldsymbol{\alpha}] \frac{-x_0}{U - \Delta} + (c^+ + c^-) \int_{\frac{-x_0}{U-\Delta}}^{\frac{-x_0}{\beta U - \Delta}} \int_{\beta}^{Z(s)} \alpha f(\alpha) d\alpha ds.$$

Since $\psi(t)$ satisfies the transversality condition (12) and the optimality conditions (9)(i)–(iii) over the entire production horizon, the proposed control is optimal for this case. □

Proof of Lemma 4.6

From Lemma 4.5 and Corollary 4.1, there are only three candidate control regimes: Start off with no production and then from a point $t_1 > 0$ produce at full rate (general sequence); always produce at full rate; or do not produce at all over the entire production horizon.

First, consider the general sequence having some known $t_1 > 0$. Thus

$$u(t) = \begin{cases} 0 & \text{for } 0 \le t \le t_1 \\ U & \text{for } t_1 < t \le T \end{cases}$$

which implies

$$Y(t) = \begin{cases} 0, & \text{for } 0 \le t \le t_1 \\ U(t - t_1), & \text{for } t_1 < t \le T \end{cases}. \tag{23}$$

Applying this control, it is not difficult to show that $Z(t)$ is nondecreasing over the entire production horizon. Define t_2 such that $Z(t_2) = 1$. It follows that $0 \le t_1 < x_0/\Delta < t_2$, and

$$\begin{array}{lll} Z(t) = -\infty, & \text{for} & 0 \le t \le t_1 \\ -\infty < Z(t) \le 0, & \text{for} & t_1 < t \le \frac{x_0}{\Delta} \\ 0 < Z(t) \le 1, & \text{for} & \frac{x_0}{\Delta} < t \le t_2 \\ 1 < Z(t), & \text{for} & t_2 < t \end{array}. \tag{24}$$

This results in two sub-cases: $t_2 < T$ and $t_2 \ge T$. Consider $t_2 < T$ first. Integrating the co-state (12), and using (24), yields

$$\begin{cases} \psi(t_1) = 0 \\ \psi(t) = \psi(t_1) + \int_{t_1}^{t} \int_{Z(t)}^{\infty} c^+ \alpha f(\alpha) d\alpha d\tau \\ \quad = \psi(t_1) + c^+ E[\boldsymbol{\alpha}](t - t_1), & \text{for } t_1 < t \le \frac{x_0}{\Delta} \\ \psi(t) = \psi\left(\frac{x_0}{\Delta}\right) + \int_{\frac{x_0}{\Delta}}^{t} \left[\int_{Z(t)}^{1} c^+ \alpha f(\alpha) d\alpha \right. \\ \qquad \left. - \int_0^{Z(t)} c^- \alpha f(\alpha) d\alpha \right] dt, & \text{for } \frac{x_0}{\Delta} < t \le t_2 \\ \psi(t) = \psi(t_2) - \int_{t_2}^{t} \int_{-\infty}^{Z(t)} c^- \alpha f(\alpha) d\alpha d\tau \\ \quad = \psi(t_2) - c^- E[\boldsymbol{\alpha}](t - t_2), & \text{for } t_2 \le t < T \\ \psi(T) = 0. \end{cases} \tag{25}$$

Solving equation array (25), using the fact that $\psi(t)$ is continuous over time, it follows that

$$\int_{\frac{x_0}{\Delta}}^{t_2} \left[\int_{\frac{t\Delta - x_0}{U(t-t_1)}}^{1} c^+ \alpha f(\alpha) d\alpha - \int_0^{\frac{t\Delta - x_0}{U(t-t_1)}} c^- \alpha f(\alpha) d\alpha \right] dt + c^+ E[\boldsymbol{\alpha}] \left(\frac{x_0}{\Delta} - t_1 \right) - c^- E[\boldsymbol{\alpha}](T - t_2) = 0. \tag{26}$$

Since $Z(t_2) = 1$, we also have

$$Z(t_2) = \frac{t_2\Delta - x_0}{Y(t_2)} = \frac{t_2\Delta - x_0}{U(t_2 - t_1)} = 1. \tag{27}$$

Solving for t_1 from (27), and plugging it into (26), we obtain the equation for t_2, i.e.,

$$c^+ E[\boldsymbol{\alpha}] \frac{t_2\Delta - x_0}{U} - c^- E[\boldsymbol{\alpha}](T - t_2) = (c^+ + c^-) \int_{\frac{x_0}{\Delta}}^{t_2} \int_0^{\frac{t\Delta - x_0}{Ut + (\Delta - U)t_2 - x_0}} \alpha f(\alpha) d\alpha dt. \tag{28}$$

Note that the order of integration can be interchanged on the right-hand side of (28). After some algebraic manipulation, the implicit equation for t_2 as stated in (16) is obtained. In addition, t_1 is determined using (17). This solution will be feasible if $t_1 > 0$ and $t_2 < T$. Also,

$$\psi(t) < 0,\ 0 \le t \le t_1;\ \psi(t) > 0,\ t_1 \le t < T \text{ and } \psi(T) = 0$$

so that the optimality conditions (9)(i)–(iii) are satisfied.

However, if $t_2 \ge T$, t_2 vanishes and the co-state variable will be nonzero in only two time intervals. Using a parallel argument as before, we obtain

$$\begin{array}{lll} u(t) = 0, Y(t) = 0, & \text{for} & 0 \le t \le t_1 \\ u(t) = U, Y(t) = U(t - t_1), & \text{for} & t_1 < t \le T \end{array} \tag{29}$$

and

$$\begin{cases} Z(t) = -\infty, & \text{for } 0 \le t \le t_1 \\ -\infty < Z(t) \le 0, & \text{for } t_1 < t \le \frac{x_0}{\Delta} \\ 0 < Z(t) \le 1, & \text{for } \frac{x_0}{\Delta} < t \le T \end{cases}. \tag{30}$$

Using (29) and (30), integration of the co-state (12) yields

$$\begin{cases} \psi(t_1) = 0 \\ \psi(t) = \psi(t_1) + \int_{t_1}^{t} \int_{Z(t)}^{\infty} c^+ \alpha f(\alpha) d\alpha d\tau \\ \quad = \psi(t_1) + c^+ E[\boldsymbol{\alpha}](t - t_1), & \text{for } t_1 < t \le \frac{x_0}{\Delta} \\ \psi(t) = \psi\left(\frac{x_0}{\Delta}\right) + \int_{\frac{x_0}{\Delta}}^{t} \left[\int_{Z(t)}^{1} c^+ \alpha f(\alpha) d\alpha \right. \\ \qquad \left. - \int_0^{Z(t)} c^- \alpha f(\alpha) d\alpha \right] dt, & \text{for } \frac{x_0}{\Delta} < t \le T \\ \psi(T) = 0 \end{cases}. \tag{31}$$

By rearranging equation array (31), (18), stated in the lemma, is obtained.

This completes the proof for the general sequence. The other two sequences are proved similarly. Specifically, if $t_1 \le 0$, the only feasible solution left which satisfies optimality conditions (9) is production at full rate as stated in the lemma (the second sequence). The third sequence can never hold, because $(x_0/\Delta) < T$ implies that the inventory level will become negative prior to time T. Thus, no production will not be optimal. This conclusion may also be obtained by showing that $\psi(t)$ cannot be negative during the entire production horizon. □

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