Real-time waiting-price trading interval in a heterogeneous options market: A Bernoulli distribution

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Abstract

Options pricing remains an open research question that is challenging for both theoreticians and practitioners. Unlike many classical binomial models that assume a "representative agent" the suggested model considers two players who are heterogeneous with respect to their underlying asset price distributions on expiration day and their levels of willingness to make a transaction (eagerness level). A two-player binomial model is developed to find the real-time optimal option price in two stages. First, we determine a primary feasible pricing domain. We then find a narrower feasible domain, termed the "waiting-price trading interval", meaning the region within which the players may wait for better offers due to a change in market conditions or player beliefs or make an immediate transaction. The suggested model is formulated by a non-linear optimization model and the optimal price is shown to be unique. We show that the counter player's eagerness level has a significant effect on the optimal option price offer. Through a numerical analysis, we conclude that the waiting-price trading interval for executing a transaction narrows as the transaction costs increase, and widens with an increase in the perceived probabilities for the underlying asset price on expiration day.

Keywords: option pricing; optimization; waiting-price trading interval; heterogeneous players

1. Introduction

1.1 Options overpricing

The role of option contracts have been widely used in the supply chain management to provide the retailer with flexibility to respond to unanticipated customer demand. Under option contract, the retailer gains the right but not the obligation to engage in the transaction. Yuan et al. (2018) analyzed a system assuming a newsvendor-like retailer who orders call option from the supplier with an emergency order opportunity. In options contracts, the option price represents the amount of money that a buyer pays the seller for having the opportunity to buy the underlying asset at the strike price. The market prices of options tend to exceed the option prices predicted by existing theoretical models, a phenomenon known in the literature as "over-pricing" (Bakshi and Kapadia, 2003). Coval and Shumway (2001) showed that option prices according to the S&P500 index exceed the prices evaluated by classical models (such as Black and Scholes, 1973; Merton, 1973) due to factors not included in the classical models, such as the volatility risk associated with underlying asset prices and high transaction costs.

Empirical evidence for "overpriced" options has been provided in a number of previous studies dating back several decades (see, for example, Jackwerth, 2000; Bakshi, Cao, and Chen, 1997; and Broadie and Detemple, 2004). More recently, by comparing historical option returns with the returns generated using option pricing models, and using a sample from 1987-2012 on the S&P500 index, Chambers et al. (2014) showed the existence of a pricing gap between the theoretical price and the market price of put options. They found that returns on selling put options are not consistent with the theoretical options' prices and that the average cost of buying out-of-the-money put options to add tail-risk protection to a portfolio may include a significant premium. Faias and Santa-Clara (2017) proposed a method to maximize the expected utility of a portfolio of European options, held to maturity. They compared the current price of an option with the expected payoff of the option on its expiration date given the assumed probability distribution of the underlying asset implicit in the simulation. If the option appears cheap (expensive) relative to the expected payoff on expiration day, the investor buys (sells) it. They found, based on a sample from 1996-2013 on the S&P500 index, that exploiting mispricing between options increases expected portfolio profit.

An important study in understanding causes of overpricing is that of Pena et al. (1999) who examined the overpricing of call and put options with theoretical prices predicted by the Black and Scholes (1973) model (hereafter the B-S model). They found that the main explanation for option overpricing is associated with high transaction costs and high bid-ask spread, especially in the case of out-of-the-money options. Because of option mispricing, using common models might lead buyers to avoid a transaction (in the case of an underestimation) or to agree to pay a higher price (in the case of overestimation). Similarly, sellers may expect higher prices (in the case of overestimation), thus reducing tradability or generating losses for numerous players (buyers) who base their decisions upon classical models.

Little attention has been paid to understanding the effect of heterogeneous beliefs on overpricing and other measures such as expected profit gained by speculators. The mismatch between theoretical and market option prices is strongly related to the general problem of evaluating options' prices. The solution to this research problem seems challenging, as options pricing is also related to the underlying market activity. Detemple and Selden (1991) demonstrated that the underlying asset price cannot be “exogenic” to option pricing because it is itself affected by the option price. Thus, when new options are issued, there is an influence on the underlying asset - its price tends to increase because of risk-taking investors who increase the demand for call options. Friesen et al. (2012) showed empirically, using traded options in the CBOE between 2003-2006, that heterogeneous beliefs cause overpricing of options beyond the effect of demand, which is an important determinant of overpricing, as argued by Bollen and Whaley (2004). Bondarenko (2014) found that historical prices of the S&P 500 put options are excessive and incompatible with canonical asset-pricing models such as the CAPM model (Sharpe, 1964). They found that none of the proposed models can possibly explain the anomaly and postulated that the subjective beliefs of investors might provide the key. Buraschi et al. (2014) examined the heterogeneous beliefs of players regarding specific stocks on the S&P1000 index in 1996-2007. They found that heterogeneous beliefs partly explained the overpricing of options both on an index level and with respect to specific stocks. More specifically, they found that overpricing arises due to disparate estimations of different players regarding the future growth of a particular firm and of the index being traded.

Kang and Lou (2016) argued that for heterogeneous players, the call option is overpriced from the perspective of the "representative agent", meaning that all the players have the same underlying asset price distribution on expiration day. However, option overpricing in general remains a puzzle. Despite the fact that some common findings have emerged from the above studies, the precise mechanism by which heterogeneous beliefs among players leads to option overpricing has not been elucidated.

This paper focuses on a Binomial model with Bernoulli distribution with two players and develops a mathematical model for real-time option pricing. Binomial models have become widespread, valuable tools for examining a variety of economic problems, such as the illiquidity problem and the overpricing problem, mainly because of the simplicity associated with the assumption of only two states of the underlying asset price on expiration day (He and Shi, 2016).

1.2 Binomial option pricing models

The binomial model was first presented in 1979 by Cox, Ross, and Rubinstein (CRR hereafter) as a discrete simplified version of the B-S classical model. According to this approach, options prices are computed by forming expectations of their payoffs over the lattice branches. This lattice approach developed for option pricing generally assume a "representative agent". Jarrow and Rudd's (1982) model (JR) assumed an "identical to all" probability of increase and decrease in the underlying asset price on expiration day. Attempts to converge the discrete multi-period CRR model for option prices into the continuous B-S model (to increase precision and decrease computational time) were further improved by Tian (1993), who assumed that the probability of each of the two underlying asset states on expiration day depends on an exponential smoothing parameter which increases with the "moneyness". Leisen and Reimer (1996) (LR) suggested an improvement to CRR's model by increasing the rate of convergence to the B-S model using the formula of normal proximity for the two states of the underlying asset price. Since the seminal model of CRR presented in 1979, lattice approach for option pricing still remain in the focus of more recent research. Chung and Shih (2007) developed a generalized version (GCRR) of the CRR binomial model. Kim et. al. (2016) developed a generalized version of JR and Tian (1993) developed tree models.

1.3 Heterogeneous beliefs

The literature about option pricing that deals with heterogeneous beliefs is scarce. Leland (1996) argues that because derivatives have zero net supply (for every long position there is an exact equivalent short position) the average or consensus investor will neither buy nor sell options. The average investor must hold the market portfolio (or the underlying asset), which includes zero net position in derivatives. Thus options will be traded only by investors who differ from the average investor. Leland (1996) study a binomial theoretical non arbitrage model to explain why players choose to buy or sell options and attributed the cause to the different beliefs the players have for the underlying asset future price. Leland (1996) found that sellers of European call options believe that the underlying asset returns are more mean-reverting than holding the market portfolio at a portion to maximize their utility function (wealth) subject to their risk aversion ("the consensus belief"). He and Shi (2016) examined the impact of heterogeneous beliefs on market equilibrium. Using the binomial model with heterogeneous beliefs they showed that agents' wealth shares are expected to remain the same under the consensus belief at is this, although they are expected to increase under their own beliefs. Their model determines a price that is affected by the players' heterogeneous beliefs and eventually derives a single price for the “representative agent” which is the “fair price”. Our model follows the ideas presented in Leland (1996) and He and Shi (2016) regarding heterogeneous beliefs, however does not assume “consensus belief” nor no-arbitrage assumptions. In particular, the suggested model is the first that considers an interaction (or trading) between different players characterized by their heterogeneity in eagerness level to trade the option.

1.4 Contribution

In this paper, we develop a real-time pricing option model assuming that the underlying asset price is distributed with Bernoulli probability function. Assuming only two possible values at expiration day T (in each time t they may be altered by each player) enables us to obtain explicit solutions. In the first stage, the suggested model produces a primary feasible price domain for possible transactions and in the second stage narrows this domain for calculating the equilibrium price for the two players (or alternatively, determines the "waiting-price trading interval"). In section 4 we show the effect of eagerness parameters (players' heterogeneity associated with the willingness to accept a given price offered by the counter player) on the optimal prices. In section 6, we develop, based on our model, a computerized simulation tool that may be used prior to decision-making on the part of the buyer or the seller to maximize their expected profit on expiration day. Our contribution to the existing literature lies in showing:

1. That the optimal price offered to the opponent (i.e., the player's decision) does not depend on his/her exposed portfolio to the underlying asset prior to the transaction.
2. The existence, uniqueness and optimality of a domain which defines the agreed waiting-price trading interval. Empirically, we find that this interval is asymmetric between the buyer and the seller.
3. That the equilibrium price of the option as defined by the midpoint between the bid and ask prices turns out to be a special case of our model whereby the players are homogeneous in their sensitivities to the price (i.e., have the same eagerness level to complete a transaction).
4. In CRR and other binomial models, the option price is computed under the risk-neutral probability measure to avoid arbitrages. Different from the these models, we allow two agents have heterogeneous beliefs about the probability distribution of the underlying asset price dynamics, therefore the model is not arbitrage free.
5. That a necessary condition for a transaction is that the buyer of a call option is assumes a higher probability for the underlying asset price increasing than the seller ("optimistic"). This conclusion is reversed for a put option, i.e., the buyer must be more pessimistic than the seller.

2. Primary feasible domain

The distribution at observation time t (current time) of the underlying asset price $f_{i,t}(s)$ at expiration day *T*, for player i is:

$$f_{i,t}(s)=\begin{cases} p_i & s_t(1+u_t) \\ 1-p_i & s_t(1-d_t) \end{cases} \quad (1)$$

where accordingly, at expiration day *T*, only two values for the underlying price can be realized, $s_t(1+u_t)$ and $s_t(1-d_t)$, with s_t denoting the current underlying asset price. We define without loss of generality $u_t=\sigma\sqrt{\Delta t}$ and $d_t=1-\frac{1}{1+u_t}$, that is, a right-sided deviation and a left-sided deviation, respectively, where σ (as in standard binomial models, e.g., CRR, GCRR) is the standard deviation of the underlying asset price for a unit of time and Δt is the duration until the expiration day, hence $\Delta t=T-t$.

We begin by finding the feasible domain for a transaction between two rational players. We assume both players are interested in trading in the option's market and that each player's objective function involves maximizing his/her own expected profit at expiration *T*. Based on (1), the expected portfolio payoff value function U at expiration day for player A - at this stage, either the buyer or the seller - is:

$$U_A(O_{A,t}(s\ ),f_{A,t}(s\ ))=(1-p_A)O_{A,t}(s_t(1-d_t))+p_AO_{A,t}(s_t(1+u_t)) \quad (2)$$

where $O_{A,t}(s)$ is a deterministic function that represents player A's total portfolio profit on expiration day when exposed to index *S* under the realization of $s\in S$. For simplicity, we assume that the free interest rate is zero, and therefore we do not discount the payoff at expiration day in (2). We define c_t^x as the profit function (without any transaction cost) on expiration day for a European call option with strike price x at time t for selling or buying an option. Since our model does not assume a priori the identity of the players (buyer vs. seller), we refer in the following to two possible scenarios.

Starting with the case that player A buys and player B sells, a necessary condition for player A, given that she currently has a general portfolio $O_{A,t}(s)$, to buy one unit of European call option, in order to increase her expected profit function is:

$$(1-p_A)[O_{A,t}(s_t(1-d_t))+c_t^x(s_t(1-d_t))]+p_A[O_{A,t}(s_t(1+u_t))+c_t^x(s_t(1+u_t))]-k\geq$$
$$(1-p_A)O_{A,t}(s_t(1-d_t))+p_AO_{A,t}(s_t(1+u_t))$$

After simplification,

$$(1-p_A)c_t^x(s_t(1-d_t))+p_Ac_t^x(s_t(1+u_t))-k\geq 0 \quad (3)$$

Condition (3) reflects the intuitive explanation under which the expected revenue (the amount that is subjectively gained at expiration) is bigger than the transaction cost (the amount that is paid). Solving (3) as an equality for player A as the option's *buyer* (or equivalently for player B as the buyer), where the transaction cost for selling or buying one unit option is denoted by k, we obtain the maximal price for buying the given option:

$$c_{\max,\mathrm{A},t} = -k + \begin{cases} 0 & s_t(1+u_t) \le x \\ p_A(s_t(1+u_t) - x) & s_t(1-d_t) \le x < s_t(1+u_t) \\ (1-p_A)(s_t(1-d_t) - x) + p_A(s_t(1+u_t) - x) & x < s_t(1-d_t) \end{cases} \tag{4}$$

By solving (3) as equality for player A as the option's *seller* (or equivalently for player B as the seller), we obtain the minimal price for selling the given option:

$$c_{\min,\mathrm{A},t} = k + \begin{cases} 0 & s_t(1+u_t) \le x \\ p_A(s_t(1+u_t) - x) & s_t(1-d_t) \le x < s_t(1+u_t) \\ (1-p_A)(s_t(1-d_t) - x) + p_A(s_t(1+u_t) - x) & x < s_t(1-d_t) \end{cases} \tag{5}$$

Using equation pair (4) and (5), along with a similar pair of equations for player B, we are able to construct the feasible domain as $c_t^x \in [c_{\min,A,t}, c_{\max,B,t}]$ when player A sells and as $c_t^x \in [c_{\min,B,t}, c_{\max,A,t}]$ when player A buys. In these domains, transactions match both players' interests (i.e., improve both their expected profit); therefore the players' identity is revealed. Hence, in the feasible domain there may be at most one transaction where player A buys and player B sells, or vice versa. In Lemma 1, we present the necessary condition for a transaction between the two players.

Lemma 1. A necessary condition for a transaction between seller A and buyer B is:

1) For $s_t(1+u_t) \le x$, $k = 0$ and the option's price is 0.

2) For $s_t(1-d_t) \le x < s_t(1+u_t)$, $p_B \ge p_A + \dfrac{2k}{s_t(1+u_t) - x}$.

3) For $x < s_t(1-d_t)$, $p_B \ge p_A + \dfrac{2k}{s_t(d_t + u_t)}$.

Proof. The results follow from constructing the feasible domain $c_t^x \in [c_{\min,A,t}, c_{\max,B,t}]$ and by substituting (4) for player B and (5) for player A. ■

From case (1) in Lemma 1 we conclude that a transaction cannot be realized when $s_t(1+u_t) \le x$. Moreover, we conclude that when player A sells and player B buys, and the transaction cost is positive, the necessary condition for any transaction implies that player B is more optimistic than player A in terms of perceived probability, that is, $p_B > p_A$ for call options (the conclusion is reversed for put options, i.e., player B is more pessimistic than player A). Hence, when both players are homogeneous in their expectation for the underlying price on expiration day (i.e., they assume nearly the same probability for the value of the underlying price on expiration day), a transaction does not occur, unless the transaction cost is diminished.

We now visually show (see Figure 1) feasible and infeasible transactions between the buyer (player B - blue) and the seller (player A - red). Without losing generality, we again refer to the case where player A sells and player B buys. For given values of s_t, u_t, d_t, x, k, there are three feasible scenarios for player A's minimal price as the option's seller and for player B's maximal price as the option's buyer, as described in (5) and (4), respectively. Figure 1 shows the boundaries for both player A's selling price, that is, $c_{\min,A,t}$ (the red line) and player B's buying price, $c_{\max,B,t}$ (the blue line) as a function of the probability that the underlying asset price increases in ratio u_t on expiration day, and under the assumption that both players' identity is known:

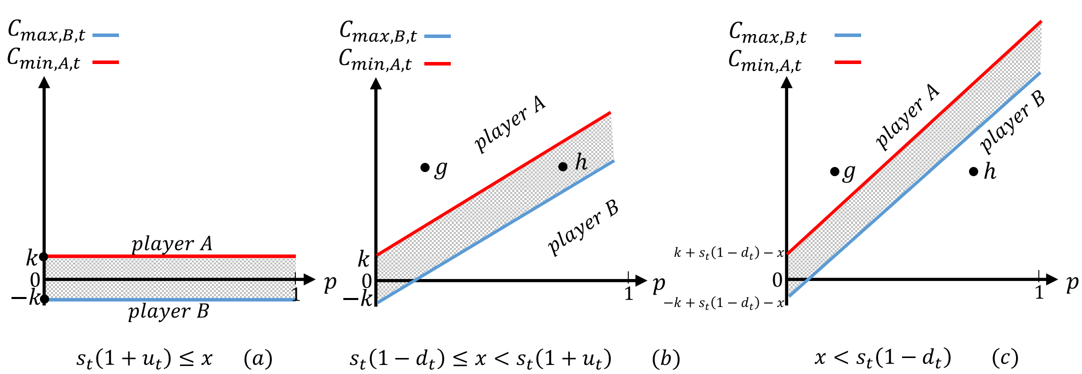

Figure 1. Maximal and minimal option prices for the buyer (player B - blue) and the seller (player A - red), respectively, as a function of perceived probability for the underlying asset price on expiration day when each player's identity is known.

Points *h* (associated with the buyer) and points *g* (associated with the seller) in Figure 1 represent theoretical transactions (i.e., matched prices) between player A and B; therefore, they have the same height. For example, in the scenario $x < s_t(1-d_t)$, these points represent a feasible price for a transaction between players A and B. However, for the scenario $s_t(1-d_t) \le x < s_t(1+u_t)$, these points represent an infeasible price, because the buying price of player B (see point *h*) is higher than the maximum price (the blue line). In conclusion, for player A as the seller and player B as the buyer, given the boundaries of the feasible domain as a function of the current underlying asset price and the perceived distributions of each player, the necessary condition on c_t^x for a transaction between the players is:

$$c_{\min,A,t} \le c_t^x \le c_{\max,B,t} = \begin{cases} k \le c_t^x \le -k & \text{if } s_t(1+u_t) \le x \\ p_A(s_t(1+u_t)-x)+k < c_t^x \le p_B(s_t(1+u_t)-x)-k & \text{if } s_t(1-d_t) \le x < s_t(1+u_t) \\ (1-p_A)(s_t(1-d_t)-x)+p_A(s_t(1+u_t)-x)+k < c_t^x \le (1-p_B)(s_t(1-d_t)-x)+p_B(s_t(1+u_t)-x)-k & \text{if } x < s_t(1-d_t) \end{cases} \quad (6)$$

In the following section we develop an additional hierarchical stage which considers players' eagerness to execute a transaction within the feasible domain or alternatively, to wait for a better price proposal by the counter player.

3. The waiting-price trading interval

Without loss of generality, we assume that both players are interested in trading one unit of a call option and that player A sells the option and player B buys it. The next stage in developing our model is to find the "waiting-price trading interval" $[l_A, l_B]$, where l_A and l_B are decision variables representing the bids offered by player A and player B (respectively) to their opponent. This new feasible domain (herein, interval) is an outcome of both players' attempts to independently maximize their own expected profit by the end of the process, while taking into account the eagerness level of the opponent to agree to a given price. In our model, each player in the trading system creates a “quote” (i.e., price offer) for the other player, obtained by solving an optimization problem described below. The result of this stage is the narrowing of the primary feasible domain into a smaller interval (the waiting-price trading interval) or its complete “disappearance” in the case where the transaction is executed (when the interval includes a single point) or in the case where the transaction does not occur (when the feasible domain is empty). The existence of a positive waiting-price trading interval means that a transaction does not occur immediately, i.e., the players are in a waiting mode whereby the market conditions can change afterwards (a change in the underlying asset price or the perceived distributions) and as a result, the primary feasible domain and/or the waiting-price trading interval may change too (i.e., a new offer may be suggested by one or both players). To simplify the modeling of the response of each counter player to the offer, we assume the following probabilities of acceptance:

$$P_B(l_A) = 1 - (\theta_A)^{\beta} \tag{7}$$

where $\theta_A \equiv \dfrac{l_A - c_{\min,A,t}}{c_{\max,B,t} - c_{\min,A,t}}$. Also,

$$P_A(l_B) = (\theta_B)^{\alpha} \tag{8}$$

where $\theta_B \equiv \dfrac{l_B - c_{\min,A,t}}{c_{\max,B,t} - c_{\min,A,t}}$. Parameters $\alpha > 0$ and $\beta > 0$ (herein “eagerness” parameters) represent the willingness (sensitivity) to accept a given price offered by the counter player on the parts of player A and B, respectively. These probabilities allow for a heterogeneous description among the two players of consenting to a given price. Larger values of β indicate player B’s stronger willingness to agree to the ask price, assuming she is the buyer. Similarly, smaller values of α indicate player A’s stronger willingness to agree to the bid price assuming he is the seller. We define:

$c_{A,t}(l_A, s)$ - the profit (or loss) for a call option at time t without a transaction cost on expiration day, for the underlying asset s, for player A as a seller with ask price l_A.

In the following optimization problem, and according to (7)-(8), the counter player may either accept the offer or reject it. Given expected profit function (2), we obtain the following objective function for player A:

$$\max_{l_A} Z = \left\{\left(1-(\theta_A)^{\beta}\right)U_A(O_{A,t}(s)+c_{A,t}(l_A,s)-k,\ f_{A,t}(s)) + (\theta_A)^{\beta}U_A(O_{A,t}(s),\ f_{A,t}(s))\right\} =$$

$$\max_{l_A} Z = \left\{\left(1-(\theta_A)^{\beta}\right)\left((1-p_A)c_{A,t}(l_A,s_t(1-d_t)) + p_A c_{A,t}(l_A,s_t(1+u_t)) - k\right)\right\} \tag{9}$$

From (9), it follows that player A's total portfolio profit on expiration day $O_{A,t}(s)$ does not affect the pricing decision, thus supporting our assumption of trading one unit. For player A as a seller, l_A is the decision variable which represents his offer to the opponent. For each of the three feasible scenarios presented earlier, we can find the optimal waiting-price trading interval $[l_A^*, l_B^*]$.

For $s_t(1+u_t) \le x$, from (9) and Lemma 1(case 1) it seems that the waiting-price trading interval is empty, meaning $[l_A^*, l_B^*] = \varnothing$. Since there is no primary feasible domain in this case, a transaction does not occur.

For $s_t(1-d_t) \le x < s_t(1+u_t)$, by definition,

$$\theta_A \equiv \frac{l_A - c_{\min,A,t}}{c_{\max,B,t} - c_{\min,A,t}} = \frac{l_A - [p_A(s_t(1+u_t)-x)+k]}{[p_B(s_t(1+u_t)-x)-k] - [p_A(s_t(1+u_t)-x)+k]}$$
$$= \frac{l_A - [p_A(s_t(1+u_t)-x)+k]}{[p_B - p_A](s_t(1+u_t)-x) - 2k}.$$

According to (9), the optimization problem for player A is thus:

$$\max_{l_A} Z = \left\{\left(1-\left(\frac{l_A - [p_A(s_t(1+u_t)-x)+k]}{[p_B-p_A](s_t(1+u_t)-x)-2k}\right)^{\beta}\right)\left(l_A - [p_A(s_t(1+u_t)-x)+k]\right)\right\} \tag{10}$$

s.t

$$c_{\min,A,t} \le l_A \le c_{\max,B,t} \tag{10.1}$$

The following propositions imply the existence of a single optimal waiting-price trading interval. Let us denote $M \equiv c_{\max,B,t} - c_{\min,A.t}$. Thus, $M = (p_B - p_A)(s_t(1+u_t)-x) - 2k$.

Proposition 1.

For $s_t(1-d_t) \le x < s_t(1+u_t)$,

$$l_A^* = \left[p_A(s_t(1+u_t)-x)+k\right] + M\left[\frac{1}{(1+\beta)}\right]^{1/\beta} \tag{11}$$

is the unique, optimal and global solution of optimization problem (10).

Proof.

See Appendix A.

Expression (11) represents the optimal ask price offered by player A as a seller. This price is obtained based upon the assumption of full information. That is, since the optimal decision $l_A^*(\beta)$ is a function of β, in order for player A to suggest an ask price, it is assumed that information about the counter player is given. In particular, the information required about player B includes her "eagerness parameter", β, and her perceived probability, p_B.

Similarly to optimization problem (9), for player B as the option's buyer we first simplify the objective function:

$$\max_{l_B} Z = \left\{(\theta_B)^\alpha U_B(O_{B,t}(s)+c_{B,t}(l_B,s)-k,\, f_{B,t}(s)) + \left(1-(\theta_B)^\alpha\right)U_B(O_{B,t}(s),\, f_{B,t}(s))\right\} =$$

$$\max_{l_B} Z = \left\{(\theta_B)^\alpha \left((1-p_B)c_{B,t}(l_B, s_t(1-d_t)) + p_B c_{B,t}(l_B, s_t(1+u_t)) - k\right)\right\} \quad (12)$$

We conclude from (9) and (12) that each players' decision regarding the optimal offer does not depend on the exposed portfolio to the underlying asset prior to the transaction. Accordingly,

$$\theta_B \equiv \frac{l_B - c_{\min,A,t}}{c_{\max,B,t} - c_{\min,A,t}} = 1 - \frac{c_{\max,B,t} - c_{\min,A,t}}{c_{\max,B,t} - c_{\min,A,t}} + \frac{l_B - c_{\min,A,t}}{c_{\max,B,t} - c_{\min,A,t}}$$

$$= 1 + \frac{l_B - c_{\max,B,t}}{c_{\max,B,t} - c_{\min,A,t}}$$

$$= 1 + \frac{l_B - [p_B(s_t(1+u_t)-x)-k]}{[p_B - p_A](s_t(1+u_t)-x)-2k} = 1 - \frac{[p_B(s_t(1+u_t)-x)-k]-l_B}{[p_B - p_A](s_t(1+u_t)-x)-2k}$$

The optimization problem for player B is:

$$\max_{l_B} Z = \left\{\left(1 - \frac{[p_B(s_t(1+u_t)-x)-k]-l_B}{[p_B - p_A](s_t(1+u_t)-x)-2k}\right)^\alpha \left([p_B(s_t(1+u_t)-x)-k]-l_B\right)\right\} \quad (13)$$

$s.t$

$$c_{\min,A,t} \le l_B \le c_{\max,B,t} \quad (13.1)$$

Proposition 2.

For $s_t(1-d_t) \le x < s_t(1+u_t)$,

$$l_B^* = [p_B(s_t(1+u_t)-x)-k] - \frac{M}{\alpha+1} \quad (14)$$

is the unique, optimal and global solution of optimization problem (13).

Proof.

See Appendix B.

Equations (11) and (14) can be rewritten in the following forms:

$$l_A^* = c_{\min,A,t} + [c_{\max,B,t} - c_{\min,A,t}]\left[\frac{1}{1+\beta}\right]^{1/\beta} \quad (15.1)$$

$$l_B^* = c_{\max,B,t} - [c_{\max,B,t} - c_{\min,A,t}]\left[\frac{1}{1+\alpha}\right] \quad (15.2)$$

A possible explanation for the incomplete symmetry between the buyer and the seller as indicated from (15) may be associated with the asymmetry of the probability functions (7)-(8) addressing the response of each player to the counter player's offer. Following (15.1), a lower bound for l_A^* offered by player A is:

$$l_A^* \geq c_{\min,A,t} + Me^{-1} \quad (16)$$

An upper bound for player B is $c_{\max,B,t}$. As is easily noticed, these bounds are independent of the counter player's eagerness parameter. Although price offers that are exterior to the optimal domain, $[l_A^*, l_B^*]$, but are within the primary domain are still feasible, and although they comply with the condition of mutual improvement (in terms of expected profits), such offers are not posed by either of the players. This can be explained by the assumption of having full information; thus each player expects from the opponent a specific response probability to a given price offer upon which his/her own expected profits are maximized.

Finally, we note that for the third case, in which $x < s_t(1-d_t)$, result (15) is also valid, as well as the global and uniqueness characteristics of the solution. It is not shown due to space considerations.

4. The effect of eagerness parameters

The following Lemma relates to a specific case in which $\alpha = \beta = 1$. This special case represents a linear and identical sensitivity to changes in the price among both players (i.e., a homogeneous eagerness level).

Lemma 2.

Define $l_{avg}^{*} \equiv \frac{c_{\max,B,t} - c_{\min,A,t}}{2}$ as the midpoint of the primary feasible domain when player A sells and player B buys. If $\alpha = \beta = 1$, a single optimal equilibrium price is obtained, exactly at the midpoint, that is,

$$l_A^* = l_B^* = l_{avg}^* \tag{17}$$

Proof.

For the cases $s_t(1-d_t) \le x < s_t(1+u_t)$ and $x < s_t(1-d_t)$, substitute $\beta = 1$ into (15.1) and $\alpha = 1$ into (15.2). Thus we obtain $l_A^* = c_{\min,A,t} + \frac{[c_{\max,B,t} - c_{\min,A,t}]}{2}$ and $l_B^* = c_{\max,B,t} - \frac{[c_{\max,B,t} - c_{\min,A,t}]}{2}$, respectively. We conclude that $l_A^* = l_B^*$ and that this value equals $\frac{[c_{\max,A,t} - c_{\min,B,t}]}{2}$.■

This result, which is not true in general, is a special case of our model, where the midpoint of the primary domain coincides with the midpoint of the waiting-price trading interval. In numerous recent studies, the midpoint price between the bid and ask prices is commonly used as the option market price for empirical evaluatuation of pricing models (see for example, Cova and Shumway (2001), Conrad et al. (2013), Frazzini and Pedersen (2012), and Mckeon (2016)). However, in contrast with those studies, Lemma 2 assigns the same midpoint to both ranges only for the special case where $\alpha = \beta = 1$. Another difference between our model and those studies is that the bid and ask prices construct always a non-feasible domain (since the ask is always bigger than the bid), while the "waiting interval" developed in our model may be feasible (i.e., when $l_A^* \le l_B^*$). If one interprets the equilibrium price resulting from the suggested model in this case as representing both the "bid" and "ask" prices, the waiting-price trading interval in fact converges to a single point (i.e., $l_A^* = l_B^*$) and the transaction is executed without further waiting, in a similar manner to other models in recent literature. Unlike existing literature, in our model, the "waiting" possibility arises intrinsically when the players are heterogeneous regarding their eagerness level (i.e., their sensitivity to the price offered by the opponent). Since in real markets, the bid-ask midpoint price does not coincide with the actual transaction price, we conclude that in real markets players are heterogeneous and act according to their own eagerness level (i.e., markets having $\alpha = \beta = 1$ are rarely seen). Note that there are other values of α and β besides 1 that also result in $l_A^* = l_B^* = l_{avg}^*$.

Lemma 3.1 and Lemma 3.2 below postulate that the optimal price offered increases with the eagerness level of the counter player. According to (7) and (8), the higher the eagerness parameter of the seller (player A), the "tougher" he is in terms of his willingness to accept an offer (for player A, as α is reduced the player is more willing to sell). Meanwhile, the higher the eagerness parameter of the buyer (B), the more willing she is to compromise and accept an offer. These lemmas are valid for both cases, that is for $s_t(1-d_t) \le x < s_t(1+u_t)$ and $x < s_t(1-d_t)$.

Lemma 3.1.

For buyer A, if $k > 0$ then $\partial l_A^* / \partial \beta > 0$.

Proof.

By matching (15.2) to buyer A and finding the first derivative of $l_A^*(\beta)$, we have $\partial l_A^* / \partial \beta = \frac{[c_{\max,A,t} - c_{\min,B,t}]}{(1+\beta)^2}$. Since $c_{\max,A,t} - c_{\min,B,t} > 0$, for $k > 0$, we obtain $\partial l_A^* / \partial \beta > 0$. ■

Lemma 3.2.

For seller A, if $k > 0$ then $\partial l_A^* / \partial \beta > 0$.

Proof.

By (15.1), the first derivative of $l_A^*(\beta)$, we have

$$\partial l_A^* \big/ \partial \beta = \frac{\left[-[c_{\max,A,t} - c_{\min,B,t}] \left(\frac{1}{1+\beta} \right)^{\frac{1+\beta}{\beta}} \right] \left[\beta + (\beta+1)\log\left(\frac{1}{1+\beta} \right) \right]}{\beta^2}$$

. Since $\beta > 0$, the value in the right-hand brackets is negative, and since $c_{\max,A,t} - c_{\min,B,t} > 0$, for $k > 0$, the value in the left-hand brackets is also negative, so $\partial l_A^* \big/ \partial \beta > 0$. ■

The intuitive explanation for both results in Lemma 3 is as follows. As the counter player becomes "tougher" in terms of willingness to consent to the transaction, the price offered by the buyer increases and the price suggested by the seller decreases. Although this result is not surprising, it indicates that the logic of the proposed model is sound and applies to the real options market. Theorem 1 presents sufficient conditions for the existence of an immediate transaction between seller A and buyer B.

Theorem 1.

When player A sells the option and player B buys it, the necessary conditions for an immediate transaction are:

$$\alpha = \frac{\left(\frac{1}{1+\beta} \right)^{1/\beta}}{1 - \left(\frac{1}{1+\beta} \right)^{1/\beta}} \text{ and } c_{\max,B,t} \geq c_{\min,A,t} \tag{18}$$

Proof.

The condition for an immediate transaction when player A sells and player B buys is $l_B^*(\alpha) = l_A^*(\beta)$ (for both cases: $s_t(1-d_t) \leq x < s_t(1+u_t)$ and $x < s_t(1-d_t)$), that is

$$M \left[1 - \left(\frac{1}{1+\alpha} \right) - \left(\frac{1}{1+\beta} \right)^{1/\beta} \right] = 0 \tag{19}$$

From the condition on the existence of the primary feasible domain, shown in (6), for both cases, $s_t(1-d_t) \leq x < s_t(1+u_t)$ and $x < s_t(1-d_t)$, $c_{\max,B,t} \geq c_{\min,A,t}$; thus $M \geq 0$ by definition. As a result, equality (19) is valid only if the second multiplier equals zero.

This condition, $1-\left[\frac{1}{1+\alpha}\right]-\left[\frac{1}{1+\beta}\right]^{1/\beta}=0$, is equivalent to the condition presented in the theorem under $\beta>0$. ■

Figure 2 shows the sub-domain of pairs (α,β) of eagerness parameters under which condition (18) is valid (the blue line) for $c_{\max,B,t}\geq c_{\min,A,t}$. The blue curve represents the threshold which an immediate transaction, with price $l_B^*=l_A^*$, occurs between seller A and buyer B (because $(l_B^*=l_A^*)$). The domain below the curve indicates pairs (α,β) under which a transaction does not occur. The domain above the curve indicates pairs (α,β) under which a transaction does not immediately occur, that is, the players wait for better offers (because $l_B^*>l_A^*$). We observe from Figure 2 that the domain, including "immediate transaction" and "waiting") is significantly bigger than the domain of "no transaction". Surprisingly to some, in markets where the sellers are "tougher" (i.e., have higher values of α), tradability increases, and vice versa.

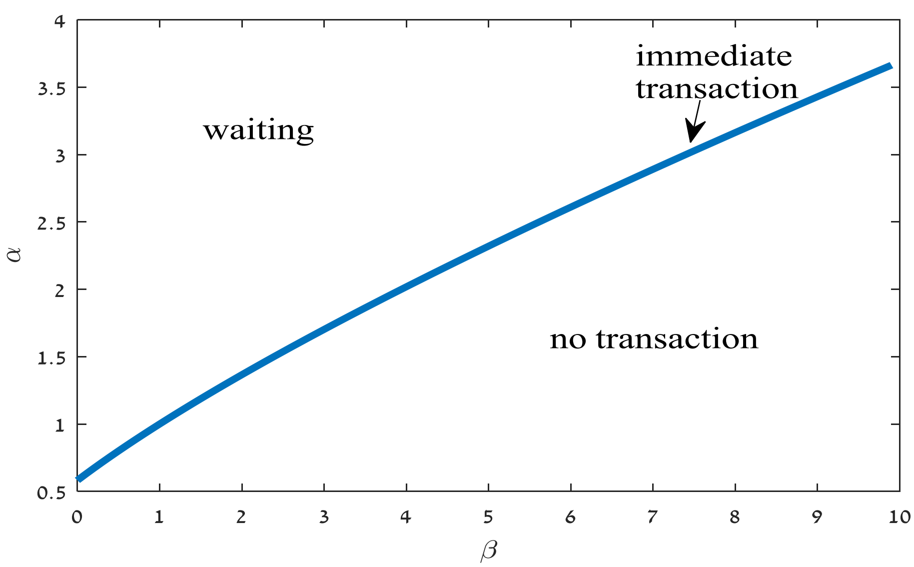

Figure 2. Domains of eagerness parameters (α,β) for the two possible scenarios - an immediate transaction and a waiting mode - when player A sells and player B buys.

5. Numerical example

5.1. Application of the model and its implications

In this section, we present a numerical analysis for the model discussed above. Specifically, we find the option's price and carry out a sensitivity analysis for the key parameters. We refer in this example to the following input parameters:

$s_t = 100, u_t = 0.1, d_t = 0.0909, k = 0.01, p_A = 0.4, p_B = 0.6$.

Exemplifying $s_t(1-d_t) \le x < s_t(1+u_t)$ we assign $x = 100$. According to the first stage of our approach, we present the feasible domain for possible transactions between the two players. According to (4), if A is assumed to be the buyer, then $c_{\max,\mathrm{A},t} = 3.99$, while if B is assumed to be the buyer then $c_{\max,B,t} = 5.99$. Similarly, for the cases seller A and seller B, according to (5), $c_{\min,\mathrm{A},t} = 4.01$ and $c_{\min,B,t} = 6.01$, respectively. Hence, the identity of the players (buyer or seller) is set endogenously, i.e., player A sells the option and player B buys it, and the feasible primary domain for the option's price is $\mathrm{c}_t \in [4.01, 5.99]$. In the next stage, the waiting-price trading interval is computed according to the above values. Figure 3 presents the optimal value for each player’s offer, as a function of the eagerness parameter α, β of the counter player, according to (15). The curves in Figure 3 are valid for each eagerness parameter, which is exogenous to the model. In the special case where $\alpha = \beta = 1$, the equilibrium price equals 5 (i.e., the midpoint) and an immediate transaction is carried out. With reference to a specific price (horizontal line) we are able to classify different populations of buyers and sellers. For example, consider the upper dotted price line, which contains the points $l_A^* = l_B^* = 5.42$. It can be seen that when the buyer has an eagerness parameter (β) less than 5.5, it causes the seller (player A) to agree to a price lower than 5.42. Similarly, if the seller has an eagerness parameter (α) greater than 2.42, where higher values of α mean that he is less willing to sell, then the buyer (player B) is convinced to accept a price higher than 5.42.

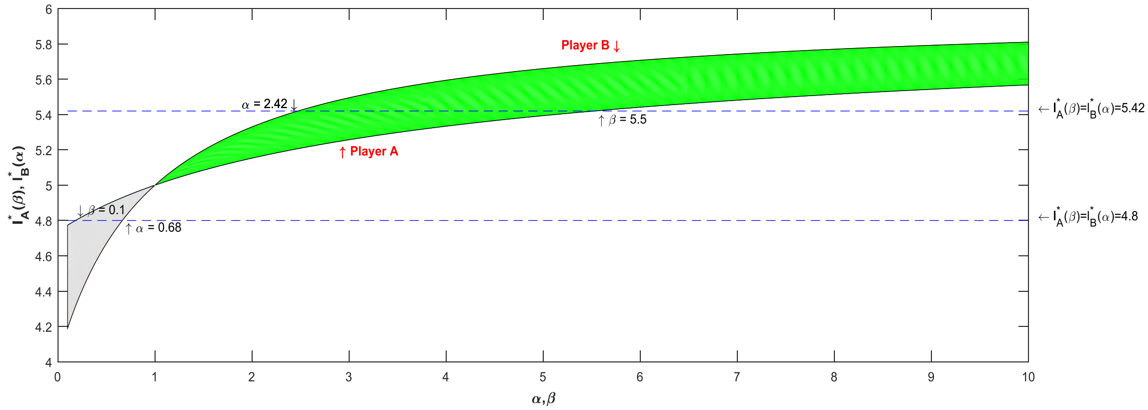

Figure 3. Optimal offer made by each player as a function of the eagerness parameter of the counter player, within the feasible domain $c_t \in [4.01, 5.99]$.

Obviously, if buyers with an eagerness parameter greater than 5.5 face sellers with an eagerness parameter lower than 2.42 the transaction is not executed. For reference to the lower dotted price line, which contains the points where $l_A^* = l_B^* = 4.8$, we determine that buyers with an eagerness parameter lower than 0.1 cause sellers to accept a price lower than 4.8. An immediate transaction (or "waiting") also takes place when buyers with an eagerness parameter lower than 0.1 receive offers from sellers with an eagerness parameter greater than 0.68, because this causes buyers to accept a price higher than 4.8. Therefore, in the shaded grey domain in Figure 3, a transaction at a given price (or "waiting") occurs if the eagerness parameter of player A (seller) is higher than the eagerness parameter of player B (buyer), as opposed to in the shaded green domain, where a transaction (or "waiting") takes place if the eagerness parameter of player A (seller) is lower than the eagerness parameter of player B (buyer). The indifference point between the players occurs when $l_A^* = l_B^* = 5$. Equivalent results (excluding absolute values of the prices themselves) are obtained for the case of $x < s_t(1-d_t)$.

5.2 Option's price sensitivity

In this section, we present sensitivity analysis of several key parameters that affect the feasible primary domain of the price and the waiting-price trading interval, namely the transaction cost, the ratio of increase in the underlying asset price on expiration day, and the counter player's perceived probability for the underlying asset price on expiration day. The waiting-price trading intervals are calculated for eagerness parameters $\alpha = 5, \beta = 5$ (green) and $\alpha = 0.9, \beta = 0.3$ (red).

5.2.1 Transaction cost

Figure 4 shows the primary feasible domain (grey) and the waiting-price trading intervals (green and red) as a function of transaction cost of buying or selling one option. According to Lemma 1 (case 2), the maximal transaction costs such that a transaction between player A (seller) and player B (buyer) may still be executed, is $k = 1$ for the parameters used in this numerical example. Hence, we assume that the range of the transaction cost is [0,1). Figure 4 shows that as the transaction cost increases, both the primary feasible domain and the waiting-price trading interval (both the green and red domains) become narrower. At point $k = 1$, the domain converges to a single point (non-feasible).

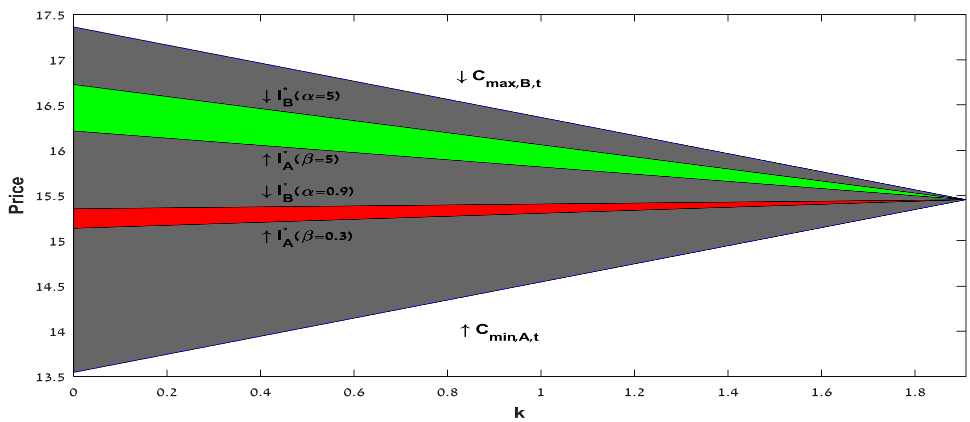

Figure 4. Feasible domains for the transaction price as a function of transaction cost for the case $s_t(1-d_t) \le x < s_t(1+u_t)$.

According to (6), as the transaction cost increases, the primary feasible domain narrows symmetrically (see also (4), (5)). The intuitive explanation is that the minimal price that seller A is willing to accept increases as the transaction cost increases, while meaning the buyer B decreases the maximal price she is willing to pay as the transaction cost increases. According to (11) and (14), the interval length decreases linearly with the transaction cost. As explained for the primary feasible domain, an intuitive explanation is that rational players wish to maximize their profit and be compensated for an increased transaction cost. As transaction costs increase, both players' profits are eroded; however the two players respond differently from each other. When $\alpha = 5, \beta = 5$, both the buyer's bid and the seller's ask price decrease as the transaction cost increases (see the green shaded domain). Thus in this case, the seller is more willing than the buyer to "absorb" the increased cost of the transaction. In the red domain, however ($\alpha = 0.9, \beta = 0.3$), both slopes *increase* with transaction cost, meaning that the buyer is more willing than the seller to "absorb" the increased transaction cost. We link both sets of behaviors to the position of each of the shaded domains. Specifically, Figure 4 shows that the green shaded domain is closer to the buyer's line (of the primary domain) than to the seller's. Under the given eagerness parameters $\alpha = \beta = 5$, the buyer is very "compromising" and the seller is very "tough". The red domain is closer to the seller's line (of the primary domain) than to the buyer's. Under the given eagerness parameters, $\alpha = 0.9, \beta = 0.3$, the buyer is "tough" and the seller is a "compromiser". Equivalent results emerge for the case $x < s_t(1-d_t)$. We conclude that decreasing exogenous costs (e.g., transaction costs) increases the probability of tradability between the players (i.e., wider feasible domain).

5.2.2 The ratio of increase in the underlying asset price on expiration day

We chose to exemplify the case $x < s_t(1-d_t)$ as it is more insightful than the other case. Figure 5 shows that the primary feasible domain (grey) as well as the waiting-price trading intervals (green and red) increase with the ratio of increase in the underlying asset price on the expiration day. According to Lemma 1, for the case $x < s_t(1-d_t)$, using the parameters specified for this numerical example, there exists a lower bound at 0.0005 for the minimal ratio of increase under which a transaction can be executed when player A sells and player B buys. Therefore, the presented ratio of increase is within the range $u_t = (0.0005, 1]$.

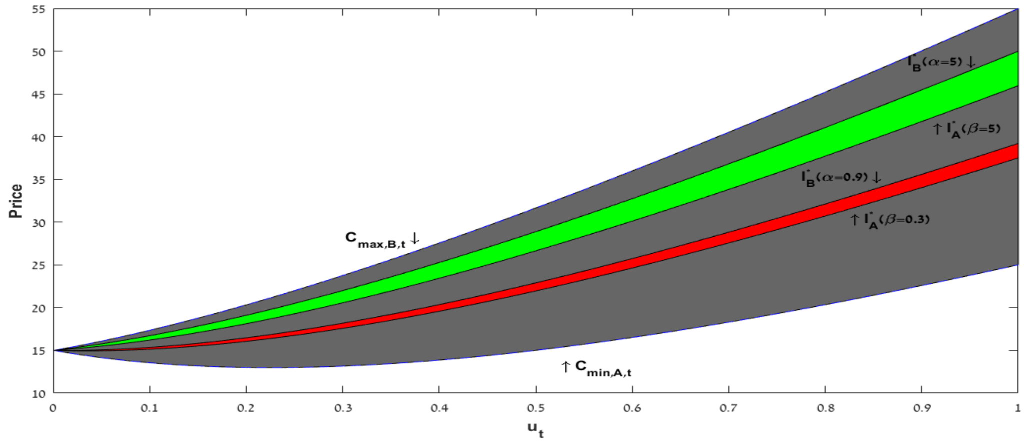

Figure 5. Feasible price domains of both players as function of the ratio of increase in the underlying asset price on expiration day for the case $x < s_t(1-d_t)$.

According to (6) and (15), respectively, and as illustrated in Figure 5, the higher the ratio of increase, u_t, the larger both the primary feasible domain and the waiting-price trading interval. As the role of u_t in these expressions is opposite to that of the transaction cost *k*, similar arguments provided for the sensitivity regarding the transaction cost (i.e., Figure 4) are applied here. Considering the fact that under the example settings the buyer is more optimistic than the seller, i.e., $p_B > p_A$, the option's expected payoff on expiration day increases with u_t in the eyes of the buyer; thus she is willing to offer higher prices. Similarly, as the seller is more pessimistic, he increases his offer to compensate for what he perceives as an increased future risk. Under the given eagerness parameters $\alpha = \beta = 5$, the buyer is very "compromising" and the seller is very "tough"; thus, similarly to Figure 4, the waiting-price trading interval is located closer to the line for the buyer.

Interestingly, Figure 5 shows that the boundaries of the feasible domain are convex functions of the ratio of increase u_t, which can be explained by the non-linear shape of $d_t = 1 - \dfrac{1}{1+u_t}$. Similar results are obtained for the case $s_t(1-d_t) < x \le s_t(1+u_t)$, but with a key difference, namely that the boundaries of the primary feasible domain and the boundaries of the waiting-price trading interval are linear. Since for the case

$s_t(1-d_t) \le x < s_t(1+u_t)$ the primary feasible domain does not depend on the ratio of decrease, d_t, and is affected only by u_t, we expect a linear curve.

5.2.3 The counter player's perceived probability

For the case $s_t(1-d_t) \le x < s_t(1+u_t)$, Figure 6 shows the primary feasible domain (grey) and the waiting-price trading intervals (green and red) as a function of the counter player's perceived probability for the underlying asset price on expiration day. Following (5), the lower boundary of the initial ask price proposed by player A is constant and does not depend on the counter player's probability. On the contrary, as buyer B's probability p_B increases (i.e., she becomes more optimistic), the upper boundary for the bid price increases according to (4). According to Lemma 1, for the case $s_t(1-d_t) \le x < s_t(1+u_t)$, the minimum counter player's perceived probability for executing a transaction when player A sells and player B buys is $p_B > 0.402$. Therefore we present the counter player's perceived probability within the range $p_B = (0.402, 1]$. Figure 6 shows the primary feasible domain and the waiting-price trading interval within these values. The results are similar to Figure 5 because an increase in the perceived probability indicates the buyer's increased optimism.

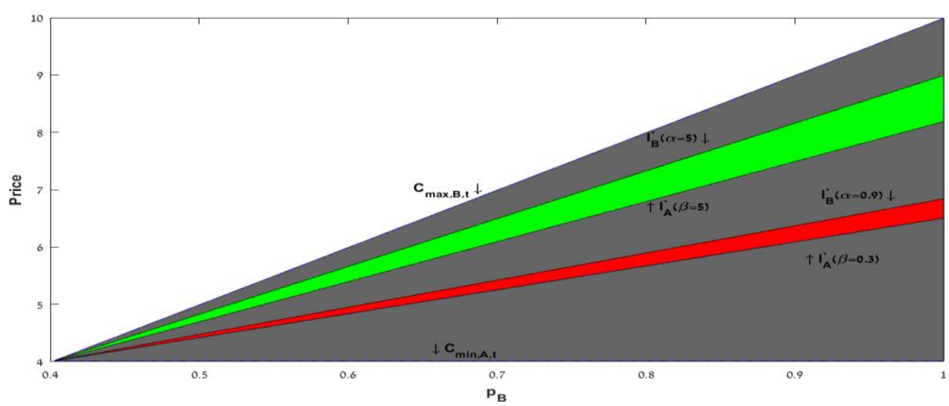

Figure 6. Feasible domain as a function of player B's perceived probability for the underlying asset price on expiration day for the case $s_t(1-d_t) < x \le s_t(1+u_t)$.

7. Concluding remarks

7.1 Conclusion

In this paper, we developed a hierarchical methodology for derivation of the explicit optimal waiting-price trading interval and proved that the boundaries of this interval are unique. The necessary conditions for an immediate transaction between seller A and buyer B were obtained. We assumed that the trading market consists of two players, each of whom maximizes his/her own perceived expected profits. We further assumed that the information upon which they base their decisions is complete and that the underlying asset price on expiration day is Bernoulli distributed. It appears that the feasible pricing interval, which narrows until an actual transaction takes place, better represents real-time trading markets than classical models, such as the models of Tian, CRR and Kang and Luo (2016), who suggested that an equilibrium price exists regardless of the market conditions or players' beliefs.

Several theoretical conclusions arise from our methodology. First, each player’s decision regarding whether or not to execute the transaction of buying or selling the option does not depend upon the exposed portfolio to the underlying asset prior to execution. This, of course, significantly reduces the complexity of the model's analysis. Second, a necessary condition (but insufficient) for carrying out the transaction is that the buyer of a call option expresses higher optimism (in terms of her estimation of the likelihood that the underlying asset price will increase) than the seller. Third, we obtained a formula for the lower bound on the seller's ask price that is independent of the counter player's eagerness level. Last, we found that the equilibrium price, defined as the midpoint of both feasible transaction domains, is obtained when the eagerness parameters are $\alpha = \beta = 1$, i.e., in the homogeneous case where both players are “indifferent” in their sensitivities. This approach differs from numerous studies that do not consider that players have heterogeneous beliefs. In those studies, the midpoint price between the bid and ask prices is commonly used as the option market price for empirical evaluation of pricing models. In contrast, according to our methodology, the bid and ask prices construct always a non-feasible domain (since the ask is always bigger than the bid), while the “waiting interval” developed in our model, may be feasible (i.e., when $l_A^* \le l_B^*$). An important finding of the second stage of the model is that it explains the real-life practice under which offers outside the feasible waiting-time trading interval $[l_A^*, l_B^*]$, although they may be theoretically possible and may even improve mutual expected profit functions, will not be posed by the players.

Further insights were obtained through numerical analysis. First, we showed that the counter player's eagerness parameter significantly affects the players' offers. The higher the counter player's eagerness parameter, and the higher the ratio of increase on expiration day, the higher the player's optimal offer. In line with common sense, the optimal option price offered by the seller lowers when facing a "tougher" player (a buyer with lower eagerness parameter), while the buyer offers a higher bid price when facing a "tougher" player (seller with higher eagerness parameter). Therefore, the probability of executing the transaction increases when players meet "tough" opponents.

7.2 The model's advantages and practical/theoretical implications

We applied an algorithm for maximizing the expected profit according to which players in the options market can subjectively decide whether or not to execute the actions of buying or selling. This represents a departure from existing models, which assume the existence of a "representative player" - therefore a single price is always presented. Rather, we demonstrate that using the proposed model, it would be preferable (in terms of increasing one's profits) to confront a compromising market when acting as the buyer, and a tougher market when acting as the seller. Although we can explain the mechanism for these outcomes, they were not expected.

Through empirical study, several binomial models for option pricing were compared. The obtained results show significant advantages to the S-H model in terms of the expected profit on expiration day. The suggested model, in contrast with previous models, considers the characteristics of a heterogeneous market and players who are heterogeneous in their eagerness level to execute the transaction.

Our model has practical implications. In markets where "tougher" buyers are expected to trade (e.g., during the days before the options expire, in times when the underlying market volatility increases dramatically, or in a bullish market), the option's seller should attempt to trace a “tough” buyer (by trying to quantify their eagerness parameter), hence generating profits from “temporary” changes in heterogeneous players' beliefs. This research has important implications for regulators who are in charge of the financial stability of the markets. In particular and contrary to expectations, we found that decreasing the transaction cost increases the probability of executing a transaction, thus increases the trading volume. The regulator determines the transaction costs and therefore has the ability to directly influence the trading volume.

7.3. Limitations and further study

We conclude that each player’s optimal price does not depend on the existing portfolio containing the underlying asset prior to executing the transaction. However, the information about the counter player, when player A is required to offer a bid, includes his/her subjective assessment of the probability of an increase in the underlying asset price on the expiration day p_B and the “eagerness” parameter β of the counter player B - information that player A does not have. Nevertheless, it is sufficient for the offering player to know only the gap between the perceived probabilities (see (11) and (14)) rather than their absolute values. Moreover, from the numerical examples, we showed that it would be sufficient to know only limited information about the opponent’s eagerness level (i.e., whether it exceeds a specified threshold or not).

There are numerous topics for further research. Among them are: applying this model's performance assessment to real trading data and comparing to other models; expanding the underlying asset scenarios on expiration day from a Bernoulli model to a continuous model, which better characterizes the world's trading systems; developing a model with incomplete information; analyzing option pricing with multiple players and other types of option styles (e.g., American, Asian); and enabling different targets for different players.

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Appendix A

Define $g(l_A) \equiv l_A - \left[p_A(s_t(1+u_t)-x)+k \right]$. Problem (10) becomes:

$$\max_{l_A} Z = \left\{ \left(1 - \left(\frac{g(l_A)}{M} \right)^{\beta} \right) g(l_A) \right\} \tag{A.1}$$

$s.t$

$$p_A(s_t(1+u_t)-x)+k \le l_A \le p_B(s_t(1+u_t)-x)-k \tag{A.2}$$

In a first step, we show the concavity of the objective function and in a second step, we show that the solution of the necessary condition equation for maximizing the reduced problem (i.e., without constraint (a.2)) satisfies the constraint.

$$\partial Z(l_A) \big/ \partial l_A = \left(\left(1 - \left(\frac{g(l_A)}{M} \right)^{\beta} \right) g'(l_A) - \frac{\beta}{M} \left(\frac{g(l_A)}{M} \right)^{\beta-1} g'(l_A) g(l_A) \right)$$

Since $g'(l_A) = 1$ the FOC becomes

$$1 - (1+\beta) \left(\frac{g(l_A)}{M} \right)^{\beta} = 0 \tag{A.3}$$

Based on Lemma 1, $p_B > p_A$ is satisfied, hence $M \ge 0$ is satisfied. Therefore,

$$\partial^2 Z(l_A) \big/ \partial {l_A}^2 = -(1+\beta) \frac{\beta}{M} \left(\frac{g(l_A)}{M} \right)^{\beta-1} g'(l_A) < 0 \, .$$

Since the second derivative is negative, the objective function is strictly concave with a single maximum. On the left-hand side, $l_A = p_A(s_t(1+u_t)-x)+k$, i.e., $g(l_A) = 0$ with a value of $Z(l_A) = 0$ and derivative $\partial Z(l_A) \big/ \partial l_A = 1$. On the right-hand side $l_A = p_B(s_t(1+u_t)-x)-k$. By definition, $l_A = c_{\max,B,t}$, $\frac{g(l_A)}{M} = 1$ with the value $Z(l_A) = 0$ and derivative $\partial Z(l_A) \big/ \partial l_A = -\beta$. In conclusion, the objective starts at the left edge of the feasible domain from the value of 0, with a positive derivative, and reaches the value of 0 again at the right-hand edge with a negative derivative. Considering the strict concavity, a single solution must satisfy FOC (a.3), which is included in the feasible domain and is also the global solution for problem (10). Solving problem (a.1-a.2) is obtained by solving equation (a.3):

$$g(l_A) = M\left[\frac{1}{(1+\beta)}\right]^{1/\beta}$$

After substitution of $g(l_A)$ we obtain (11), or in more detail,

$$l_A^* = \left[p_A(s_t(1+u_t)-x)+k\right]+\{\left[p_B-p_A\right](s_t(1+u_t)-x)-2k\}\left[\frac{1}{(1+\beta)}\right]^{1/\beta} \qquad \text{(A.4)}$$

Solution (a.4) is feasible, i.e., it satisfies constraint (a.2) since $\beta > 0$, $p_B \geq p_A$ and hence the validity of proposition 1 is demonstrated. ■

Appendix B

Define $g(l_B) \equiv [p_B(s_t(1+u_t)-x)-k]-l_B$. Problem (13) becomes:

$$\max_{l_B} Z = \left\{\left(1-\frac{g(l_B)}{M}\right)^{\alpha} g(l_B)\right\} \qquad \text{(B.1)}$$

$$s.t$$

$$p_A(s_t(1+u_t)-x)+k \leq l_B \leq p_B(s_t(1+u_t)-x)-k \qquad \text{(B.2)}$$

In a first step, we solve the problem without constraint (b.2). The first derivative of the objective function is

$$\partial Z(l_B)/\partial l_B = \left(\left(1-\frac{g(l_B)}{M}\right)^{\alpha} g'(l_B) - \frac{\alpha}{M}\left(1-\frac{g(l_B)}{M}\right)^{\alpha-1} g'(l_B)g(l_B)\right)$$

Since $g'(l_B) = -1$ the FOC becomes:

$$\left(1-\frac{g(l_B)}{M}\right)^{\alpha-1}\left(\frac{(\alpha+1)g(l_B)}{M}-1\right)=0 \qquad \text{(B.3)}$$

This equation has two solutions. The first is when $g(l_B) = M$, i.e., $l_B^* = c_{\min,A,t}$. This solution is possible and by substituting it into the objective function we obtain $Z(l_B) = 0$. The second solution to equation (b.3) is

$$l_B^* = [p_B(s_t(1+u_t)-x)-k]-\frac{M}{\alpha+1} \qquad \text{(B.4)}$$

After substituting M , we obtain (14), or in more detail,

$$l_B^* = [p_B(s_t(1+u_t)-x)-k] - \frac{[p_B - p_A](s_t(1+u_t)-x)-2k}{\alpha+1} \qquad \text{(B.5)}$$

Solution (b.5) is feasible, i.e., it satisfies (b.2), for $\alpha > 0$. Now, we only need to show that the objective value satisfies $Z(l_B^*) > 0$ and it is the maximum point. The second derivative of the objective function is:

$$\partial^2 Z(l_B) \Big/ \partial {l_B}^2 = \left(1 - \frac{g(l_B)}{M}\right)^{\alpha-2} \left(\frac{(\alpha^2+\alpha)g(l_B) - 2\alpha M}{M^2}\right). \qquad \text{(B.6)}$$

We insert l_B^* into (b.6), and verify the derivative's sign:

$$\partial^2 Z(l_B) \Big/ \partial {l_B}^2 \Big|_{l_B^*} = \left(1 - \frac{g(l_B^*)}{M}\right)^{\alpha-2} \left(\frac{(\alpha^2+\alpha)g(l_B^*) - 2\alpha M}{M^2}\right)$$

To show that the second derivative is negative, we only need to show that $\frac{(\alpha^2+\alpha)g(l_B^*) - 2\alpha M}{M^2} < 0$

We substitute the solution of (b.3) into (b.6), i.e., the solution which satisfies $(\alpha+1)g(l_B^*) = M$. We need to show that $\frac{(\alpha^2+\alpha)\left(M \big/ \alpha+1\right) - 2\alpha M}{M^2} < 0$, or $\frac{-\alpha M}{M^2} < 0$. We see that $\partial^2 Z(l_B) \Big/ \partial {l_B}^2 \Big|_{l_B^*} < 0$. Since $Z(l_B) > 0$ for every internal point within the feasible domain and thus also in the special case of point l_B^*, we conclude that solution (14) is a unique and global maximum. ■