2) $$A_1[2\times 2]=\begin{bmatrix}0 & 1\\ -3 & -2\end{bmatrix}$$
$\det\,[sI-A_1]=s^2+2s+3$ (stable).

3) $$b_1=\begin{bmatrix}-1\\ 4+2\sqrt{6}\end{bmatrix};\ c_1=\begin{bmatrix}1\\ 0\end{bmatrix};\ d_1=1$$
Then

$$\left(A_1-\frac{1}{d_1}b_1c_1^T\right)A_1=\begin{bmatrix}-3 & -1\\ 6 & -(3+2\sqrt{6})\end{bmatrix} \tag{16}$$

whose eigenvalues are the zeros of $s^2+(6+2\sqrt{6})s+15+6\sqrt{6}$. Explicitly, there is an eigenvalue $s=-(3+\sqrt{6})$ of multiplicity 2.

Since Condition 3 of Theorem 1 is not satisfied, $H(s)$ in (14) is not SPR.

However, using Theorem 2, we have the following.

1) $\beta_0=1>0$.
2) The matrix in (16) has no real negative eigenvalues of odd multiplicity.
3) $T(s)$ has no imaginary axis poles.

All three conditions of Theorem 2 are satisfied, and it follows that $H(s)$ in (14) is PR.

Example 3: Let

$$H(s)=\frac{\beta_0s^2+\beta_1s+\beta_2}{s^3+\alpha_1s^2+\alpha_2s+\alpha_3}=\frac{s^2+2s+3}{s^3+2.5s^2+3s+3.5+2\sqrt{6}}=c^T(sI-A)^{-1}b \tag{17}$$

where

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -(3.5+2\sqrt{6}) & -3 & -2.5\end{bmatrix}\ b=\begin{bmatrix}1\\ -0.5\\ 1.25\end{bmatrix}\ c=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}. \tag{18}$$

Using Theorem 2, we have the following.

1) $\beta_0=1>0$.
2) $A_1=\begin{bmatrix}0 & 1\\ -3 & -2\end{bmatrix};\ b_1=\begin{bmatrix}-1\\ 4+2\sqrt{6}\end{bmatrix};\ c_1=\begin{bmatrix}1\\ 0\end{bmatrix};\ d_1=1/2$ and

$$\left(A_1-\frac{1}{d_1}b_1c_1^T\right)A_1=\begin{bmatrix}-3 & 0\\ 6 & -7-4\sqrt{6}\end{bmatrix}$$

whose eigenvalues are the zeros of $s^2+(10+4\sqrt{6})s+21+12\sqrt{6}$. Explicitly, there are two negative real eigenvalues of multiplicity 1 (odd): $s=-3$ and $s=-16.8$. Therefore, condition 2) of the theorem is not satisfied and $H(s)$ in (17) is not PR.

V. Conclusion

In this note, we have derived conditions for PR and SPR of an SISO LTI system. The derived conditions complement the conditions derived in [7] and are easily verifiable. Finally we note that the notion of Strict Positive Realness plays a central role in stability theory. In particular, testing the conditions of the Circle Criterion is equivalent to testing strict positive realness of a given transfer function for which spectral conditions are now available.

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Optimal Control of a Failure-Prone Machine Under Random Demand

Konstantin Kogan, Sheldon Lou, and Avi Herbon

***Abstract*—In this note, we consider a manufacturing system with random machine breakdowns, which is characterized by a fairly general probability distribution. The demand is not known during the finite planning horizon except the probability distribution of the cumulative demand at the end of the horizon. We propose a decomposition method that features a feedback control. Simulation is used to compare the average cost of the proposed method and that of the optimal solution. Our results of more than a hundred examples show that the difference between the two is less than 2.4%.**

***Index Terms*—Feedback, newsboy problems, optimal control, unreliable machines.**

I. Introduction

This note is motivated by production control problems in, say, fashion industry where accessory items must be delivered to apparel manufacturers before the selling season starts and often the total demand for such accessory items is not known when they are produced. The production control policy discussed in this note tries to reduce both the inventory and backlog costs and has close ties to two schools of problems. The first is the well known single period inventory models, which minimize the expected surplus or shortage cost at the end of the planning horizon and are frequently referred to as news-vendor or newsboy problems (see [5] and [9]). Some of these models assume

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K. Kogan and A. Herbon are with the Department of Interdisciplinary Studies—Logistics, Bar-Ilan University, Ramat-Gan 52900, Israel.

S. Lou is with the College of Business Administration, California State University San Marcos, San Marcos, CA 92096-0001 USA.

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that the uncertainty is due to random demand and random production yield, while machine breakdowns are ignored. The second is the optimal production control of manufacturing systems with stochastic interference, e.g., random yield and unreliable machines (see [2] and [11]). It tries to trace the demand as closely as possible at each time point of the planning horizon. To derive the optimal control, this school of methods often assumes that the inventory level is observable during the planning horizon, and the transition between an operational state to a breakdown state of the system is described by a continuous-time Markov chain (see, for example, [1], [4], and [6]).

Such a problem, while vital for shop floor managers, is regretfully complex when viewed with an optimization perspective. Rarely can theoretical solutions, in particular closed-form ones, be derived, except for a handful of cases with specific, say, exponentially distributed machine up and down durations over an infinite production horizon [10], [8]. The resulting optimal feedback policies are characterized by constant hedging or threshold points. Efforts were also made to seek conditions under which the hedging points still remain constant even though the exponentiality assumption is relaxed for an infinite horizon problem [3]). The system considered in this note is significantly different. It features a finite production horizon, a demand that is known only at the end of the horizon, except its probability distribution, and an unreliable machine whose state follows a general time-dependent probability density function. The goal is to minimize both the average inventory carrying cost during a finite production horizon and the surplus/shortage cost at the end of the horizon. Thus, the contributions of our note are: i) it extends the classical newsboy problem to incorporate production dynamics and failure-prone conditions; ii) the machine state is described by a rather general probability density function as shown in the next section. If the machine up and down time distributions are i.i.d., the aforementioned density function can always be constructed either analytically or numerically; iii) the production horizon is finite and thus the derived feedback policy is characterized by general, time-dependent thresholds; and iv) the solution approach which is based on the use of the Maximum principle with the aid of a heuristic procedure is also distinctive, as detailed in the following sections.

II. The Model

Consider a failure-prone machine producing a single product-type to satisfy a demand d at the end of the production horizon T. The demand is a random variable whose realization D is known only at the end of the production horizon. We assume that d is characterized by the probability density function $f_d(D)$ and cumulative distribution $F_d(D)$.

Machine state $\alpha(t)$ ($\alpha(t) = 1$ if the machine is operational and $\alpha(t) = 0$ if it is broken) is characterized at time t by a probability density function $f_{t,T}(A)$ such that the machine is up A time units (i.e., $\int_t^T \alpha(\tau)d\tau = A$) out of $T-t$ time units if $\alpha(t) = 1$. The cumulative distribution function of $f_{t,T}(A)$ is denoted $F_{t,T}(A)$. We use $\alpha(t,\xi)$ to identify the machine state for realization ξ.

Denote the set of all possible realizations over the entire production horizon $\{\xi\}$ by R and define $R(t,\xi)$ as

$$R(t,\xi) = \left\{ \xi' \middle| \xi' \in R, \text{ and } \alpha(\tau,\xi') = \alpha(\tau,\xi), \text{ for } 0 \le \tau \le t \right\}. \quad (1)$$

Thus, $R(t,\xi)$ consists of only those realizations that can take place in the future if by time t we have observed a realization ξ. We assume that the production rate $u(t,\xi)$ is bounded

$$0 \le u(\tau,\xi) \le U, \qquad \xi \in R, \quad 0 \le \tau \le T. \quad (2)$$

We also assume the nonanticipativity constraint

$$u(\tau,\xi) = u(\tau,\xi'), \text{ for all } \xi' \in R(t,\xi), \qquad 0 \le \tau \le t. \quad (3)$$

The inventory level $X(t,\xi)$ is described by the following equation:

$$X(t,\xi) = X(0) + \int_0^t \alpha(\tau,\xi)u(\tau,\xi)d\tau. \quad (4)$$

Our goal is to determine the optimal production rate $u(t,\xi)$, $0 \le t \le T$ that minimize the expected total cost

$$J = \underset{R,\{D\}}{E}\left[\int_0^T hX(\tau,\xi)d\tau + p^- \max\{0, D - X(T,\xi)\} + p^+ \max\{0, X(T,\xi) - D\}\right] \to \min \quad (5)$$

where h is the unit inventory carrying cost, p^+ and p^- are the unit surplus and shortage costs at the end of the production horizon, respectively.

III. Variational Analysis of the Problem Under Known Demand

As the first step, we assume that D is known and consider the variation of the objective function

$$\delta J = \underset{R}{E}\left[\int_0^T h\delta X(\tau,\xi)dt\right] + \frac{d}{du(t,\xi)}\underset{R}{E}\left[p^- \max\{0, D - X(T,\xi)\} + p^+ \max\{0, X(T,\xi) - D\}\right]\delta u(\tau,\xi). \quad (6)$$

The control variation δu at t is defined as

$$\delta u(\tau,\xi) = \begin{cases} \alpha(t,\xi)\delta u, & \text{if } t-\varepsilon < \tau \le t \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

for some small ε. According to (3) and (7), we have

$$\delta u(\tau,\xi) = \delta u(\tau,\xi'), \text{ for all } \xi' \in R(t,\xi), \qquad 0 \le \tau \le T. \quad (8)$$

Consequently, the influence of variation (7) and (8) on the inventory level $X(t,\xi)$ in the first order of ε is

$$\delta X(\tau,\xi') = \begin{cases} \varepsilon\alpha(t,\xi)\delta u, & \text{if } \tau > t-\varepsilon \\ 0, & \text{otherwise.} \end{cases} \quad \text{for all } \xi' \in R(t,\xi). \quad (9)$$

With respect to (7)–(9), variation (6) in the first order of ε takes the following form:

$$\delta J = \varepsilon\alpha(t,\xi)\delta u h(T-t) + \frac{d}{du(t,\xi)}\underset{R(t,\xi)}{E}\left[p^- \max\{0, D - X(T,\xi)\} + p^+ \max\{0, X(T,\xi) - D\}\right]\delta u. \quad (10)$$

Let us introduce a new (co-state) variable

$$\psi(t,\xi) = h(t-T) - \lim_{\beta(t,\xi)\to\alpha(t,\xi)}\left[\frac{1}{\varepsilon\beta(t,\xi)}\right]\frac{d}{du(t,\xi)} \times \underset{R(t,\xi)}{E}\left[p^- \max\{0, D - X(T,\xi)\} + p^+ \max\{0, X(T,\xi) - D\}\right]. \quad (11)$$

Then, (10) transforms into

$$\delta J = -\varepsilon\alpha(t,\xi)\psi(t,\xi)\delta u. \quad (12)$$

Consequently, the optimality condition, $\delta J \geq 0$, is

$$\delta J = -\varepsilon\alpha(t,\xi)\psi(t,\xi)\delta u \geq 0. \quad (13)$$

Since when the machine is down, no control can be applied, we consider the case when machine is up, i.e., $\alpha(t,\xi) = 1$. First, if the machine is idle, $u(t,\xi) = 0$, the only feasible variation is $\delta u \geq 0$ and, thus, condition (13) holds only when $\psi(t,\xi) \leq 0$. Similarly, when $u(t,\xi) = U$, only nonpositive variation of is feasible $\delta u \leq 0$, i.e, (13) holds only when $\psi(t,\xi) \geq 0$. Finally, the case of $0 < u(t,\xi) < U$ implies $\psi(t,\xi) = 0$. Thus, we summarize the optimal control for our stochastic problem, as shown in (14) at the bottom of the page.

IV. Expectation Analysis

The second term in the co-state (11)

$$L = \lim_{\beta(t,\xi)\to\alpha(t,\xi)} \left[\frac{1}{\varepsilon\beta(t,\xi)}\right] \frac{d}{du(t,\xi)} \mathop{E}_{R(t,\xi)} \Big[p^- \max\{0, D - X(T,\xi)\} + p^+ \max\{0, X(T,\xi) - D\}\Big] \quad (15)$$

involves the expectation of terminal inventories. To deal with (15), we propose the following heuristic approach. First, we derive a lower bound by minimizing the expected cost over all possible realizations without imposing the nonanticipativity condition. This implies that the control which could provide such a cost online does not always exist. Then we consider an online control at a time point t, impose nonanticipativity (which increases the expected cost found at the first step) at this time point and apply a small control variation to minimize the change in the cost function. The minimization results in a feedback policy (Section V). As shown in our simulation results (Section VI), this lower bound-guided solution method provides a very good approximation to the optimal solution.

Note, that given $X(t,\xi)$ and $\alpha(\tau,\xi)$ for $t \leq \tau \leq T$, problem (1)–(5) takes a deterministic form

$$\int_t^T hX(\tau,\xi)d\tau + p^- \max\{0, D - X(T,\xi)\} + p^+ \max\{0, X(T,\xi) - D\} \to \min \quad (16)$$

subject to

$$X(s,\xi) = X(t,\xi) + \int_t^s \alpha(\tau,\xi)u(\tau,\xi)d\tau \quad (17)$$

$$0 \leq u(\tau,\xi) \leq U, \qquad t \leq \tau \leq T. \quad (18)$$

We now solve this deterministic optimization problem. According to the maximum principle [7], the Hamiltonian

$$H(\tau,\xi) = -hX(\tau,\xi) + \eta(\tau,\xi)\alpha(\tau,\xi)u(t,\xi) \quad (19)$$

where

$$\dot{\eta}(\tau,\xi) = h; \eta(T,\xi) = \begin{cases} p^-, & \text{if } X(T,\xi) < D \\ -p^+, & \text{if } X(T,\xi) > D \\ p \in [-p^+, p^-], & \text{if } X(T,\xi) = D \end{cases} \quad (20)$$

is maximized and the admissible control $u(\tau,\xi)$ has the following form:

$$u(\tau,\xi) = \begin{cases} U, & \text{if } \eta(t,\xi) \geq 0 \text{ and } \alpha(t,\xi) = 1 \\ 0, & \text{if either } \eta(t,\xi) < 0 \text{ or } \alpha(t,\xi) = 0 \text{ or both.} \end{cases} \quad (21)$$

Note, since $h > 0$, the case of $u(\tau,\xi) = e$, $0 \leq e \leq U$ when $\eta(t,\xi) = 0$ over an interval of time is excluded as not feasible. Indeed, differentiating $\eta(t,\xi) = 0$ over the interval and taking into account (20) we find that $\dot{\eta}(\tau,\xi) = h = 0$, which cannot hold.

We further assume that $h < p^-$ and $hT > p^-$, because if $h \geq p^-$, then the production will always increase the total cost. On the other hand, if $hT \leq p^-$, then carrying inventory will always be cheaper than having backlog at the end of the horizon. As a result, the optimal control will be a special case of that for $hT > p^-$, as shown below. In addition, we introduce a new parameter, $G = T - (p^-/h)$, and assume that $X(t,\xi) < D$, otherwise the optimal control is trivial—stop production. In the following two lemmas, we derive the optimal control for two cases, $t < G$ and $t \geq G$.

Lemma 1: Assume that $t < G$ and t_2 satisfies $X(t,\xi) + U\int_{t_2}^T \alpha(\tau,\xi)d\tau = D$. If $t_2 < G$, then $u(\tau,\xi) = 0$ for $t \leq \tau < G$ and $u(\tau,\xi) = U$ for $G \leq \tau \leq T$. Otherwise, if $t_2 \geq G$, then $u(\tau,\xi) = 0$ for $t \leq \tau < t_2$ and $u(\tau,\xi) = U$ for $t_2 \leq \tau \leq T$.

Proof: Assume $t_2 \geq G$, so that there is enough time to satisfy demand $X(t,\xi) + U\int_{t_2}^T \alpha(\tau,\xi)d\tau = D$. Then, the co-state solution $\eta(\tau,\xi) = h(\tau - t_2)$ is feasible and thus according to (21) control $u(\tau,\xi) = 0$ for $t \leq \tau < t_2$ and $u(\tau,\xi) = U$ for $t_2 \leq \tau \leq T$ is optimal if $\eta(T,\xi) = h(T - t_2) \leq p^-$. The inequality is ensured by $t_2 \geq G$, as stated in the lemma. On the other hand, if $t_2 < G$, then using the same argument one can verify that the co-state solution $\eta(\tau,\xi) = h(\tau - t_1)$, $t_1 = G > 0$ is feasible and, thus, the control stated in the lemma is optimal. ■

Lemma 2: Assume that $t \geq G$ and t_2 satisfies $X(t,\xi) + U\int_{t_2}^T \alpha(\tau,\xi)d\tau = D$. If $t_2 > 0$, then $u(\tau,\xi) = 0$ for $t \leq \tau < t_2$ and $u(\tau,\xi) = U$ for $t_2 \leq \tau \leq T$. Otherwise, if $t_2 \leq 0$, then $u(\tau,\xi) = U$ for $t \leq \tau \leq T$.

Proof: The proof is similar to that of Lemma 1 and, therefore, omitted. ■

Expectation of Terminal Production Results

Let us denote the ratio $(D - X(t,\xi))/U$ as $Y(t,\xi)$. If control $u(\tau,\xi)$ is defined by Lemmas 1 and 2 the expectation in (15), $E_{R(t,\xi)}[p^- \max\{0, D - X(T,\xi)\} + p^+ \max\{0, X(T,\xi) - D\}]$, denoted by $E_{R(t,\xi)}[\bullet]$ can be calculated as follows:

$$\mathop{E}_{R(t,\xi)}[\bullet] = \int_{-\infty}^{Y(t,\xi)} p^- \left(D - (X(t,\xi) + UA_t)\right) f_{t,T}(A_t)dA_t + \int_{Y(t,\xi)}^{\infty} p \int_{t^*}^{T} (X(t,\xi) + UA_s - D) f_{t,T}(s, A_s)dA_s ds \quad (22)$$

$$u(t,\xi) = \begin{cases} U, & \text{if } \psi(t,\xi) > 0 \text{ and } \alpha(t,\xi) = 1 \\ u^*(t) \in [0,U], & \text{if } \psi(t,\xi) = 0 \text{ and } \alpha(t,\xi) = 1 \\ 0, & \text{if either } \psi(t,\xi) < 0 \text{ or } \alpha(t,\xi) = 0 \text{ or both.} \end{cases} \quad (14)$$

when $t \geq G$ and $A_s = \int_s^T \alpha(\tau,\xi)d\tau$

$$\underset{R(t,\xi)}{E}[\bullet] = \int_{-\infty}^{Y(t,\xi)} p^{-}\left(D - (X(t,\xi) + UA_G)\right) f_{G,T}(A_G) \times dA_G + \int_{Y(t,\xi)}^{\infty} p \int_{t^*}^{T} (X(t,\xi) + UA_s - D) \times f_{t,T}(s, A_s) dA_s ds \tag{23}$$

when $t < G$. In the previous equalities, the mutual probability density function at time t of being up A time units when the machine is switched on from time point s to T is $f_{t,T}(s,A)$

$$\int_{t^*}^{T} f_{t,T}(s,A)ds = f_{t,T}(A) \tag{24}$$

and the earliest switching point $t^* = \max\{t, Y(t,\xi)\}$. Note, that the second terms in (22) and (23) represent the cases when the inventory at the end of the production horizon equals the demand, which is why any cost $p \in [-p^-, p^+]$ can be assumed. Thus, taking into account that $X(t,\xi) + UA_s - D = 0$, (22) and (23) can be expressed as the follows:

$$\underset{R(t,\xi)}{E}[\bullet] = \int_{-\infty}^{Y(t,\xi)} p^{-}\left(D - (X(t,\xi) + UA)\right) f_{t,T}(A)dA + \int_{Y(t,\xi)}^{\infty} p \cdot 0 \cdot f_{t,T}(A)dA \tag{25}$$

$$\underset{R(t,\xi)}{E}[\bullet] = \int_{-\infty}^{Y(t,\xi)} p^{-}\left(D - (X(t,\xi) + UA)\right) f_{G,T}(A)dA + \int_{Y(t,\xi)}^{\infty} p \cdot 0 \cdot f_{G,T}(A)dA. \tag{26}$$

Due to the nonanticipativity (4), there is only one control $u(t,\xi)$ set up from t to $t+\varepsilon$, which may differ from that we would respond if all breakdowns were known in advance. Since the first terms in (25) and (26) represent the cases when the terminal inventory is less than the demand, a control change over time ε cannot convert shortage into a surplus. Specifically, with respect to these terms, we produce $\alpha(t,\xi)u(t,\xi)\varepsilon$ instead of $U\varepsilon$ for time ε which results in $\alpha(t,\xi)u(t,\xi)\varepsilon + U(A-\varepsilon)$. On the other hand, the second terms in (25) and (26) represent the cases when the cumulative production exactly equals the demand at the end of the planning horizon, $X(t,\xi) + UA - D = 0$, with the aid of the maximal control U applied at $t^* > t$, i.e., if we start to produce at t, $u(t,\xi)$, we will have a surplus, $\alpha(t,\xi)u(t,\xi)\varepsilon$. Thus, in the first order of ε, (25) and (26) take the following form:

$$\underset{R(t,\xi)}{E}[\bullet] = \int_{-\infty}^{Y(t,\xi)} p^{-}(D - (X(t,\xi) + \alpha(t,\xi)u(t,\xi)\varepsilon + U(A-\varepsilon)))f_{t,T}(A)dA + \int_{Y(t,\xi)}^{\infty} p^{+}(\alpha(t,\xi)u(t,\xi)\varepsilon)f_{t,T}(A)dA,\ (t \geq G) \tag{27}$$

$$\underset{R(t,\xi)}{E}[\bullet] = \int_{-\infty}^{Y(t,\xi)} p^{-}(D - (X(t,\xi) + \alpha(t,\xi)u(t,\xi)\varepsilon + UA))f_{G,T}(A)dA + \int_{Y(t,\xi)}^{\infty} p^{+}(\alpha(t,\xi)u(t,\xi)\varepsilon)f_{t,T}(A)dA\ (t < G). \tag{28}$$

V. Feedback Policy

Given expressions (27) and (28) for $E_{R(t,\xi)}[\bullet] = E_{R(t,\xi)}\left[p^- \max\{0, D - X(T,\xi)\} + p^+ \max\{0, X(T,\xi) - D\}\right]$, (15) is determined as follows:

$$L = \lim_{\beta(t,\xi)\to\alpha(t,\xi)} \left[\frac{1}{\varepsilon\beta(t,\xi)}\right] \frac{d}{du(t,\xi)} \underset{R(t,\xi)}{E}\left[p^- \max\{0, D - X(T,\xi)\} + p^+ \max\{0, X(T,\xi) - D\}\right] = \left(p^+ - (p^- + p^+)F_{t,T}(Y(t,\xi))\right)$$

when $t \geq G$ and

$$L = -p^- F_{G,T}(Y(t,\xi)) + p^+\left(1 - F_{t,T}(Y(t,\xi))\right), \text{when } \ t < G.$$

Therefore, the co-state variable (11) is

$$\psi(t,\xi) = h(t-T) - \left(p^+ - (p^- + p^+)F_{t,T}(Y(t,\xi))\right), \text{ if } t \geq G \tag{29}$$

$$\psi(t,\xi) = h(t-T) + p^- F_{G,T}(Y(t,\xi)) - p^+\left(1 - F_{t,T}(Y(t,\xi))\right), \text{ if } t > G. \tag{30}$$

Our results are summarized in the following theorem.

Theorem 1: Given (1)–(5), (22), and (23), and assume demand D is known and $hT > p^-$, the optimal feedback policy is shown in

$$u(t,\xi) = \begin{cases} U, & \text{if } X(t,\xi) < X^*(t) \text{ and } \alpha(t,\xi) = 1 \\ u^*(t), & \text{if } X(t,\xi) = X^*(t) \text{ and } \alpha(t,\xi) = 1 \\ 0, & \text{if either } X(t,\xi) > X^*(t) \text{ or } \alpha(t,\xi) = 0, \text{ or both} \end{cases}$$

where $Y^*(t) = (D - X^*(t))/U$, if $t \geq G$, then intermediate control $u^*(t) = \dot{X}^*(t)$ and threshold $X^*(t)$ satisfy

$$\frac{\partial F_{t,T}\left(Y^*(t)\right)}{\partial t} = -\frac{h}{p^- + p^+} \text{ and}$$
$$h(t-T) - \left(p^+ - (p^- + p^+)F_{t,T}(Y^*(t))\right) = 0, \text{ respectively}$$

otherwise $u^*(t)$ and $X^*(t)$ satisfy

$$h - p^- u^*(t) \frac{\partial F_{G,T}\left(Y^*(t)\right)}{\partial\left(Y^*(t)\right)} + p^+ \frac{\partial F_{t,T}\left(Y^*(t)\right)}{\partial t} = 0 \text{ and}$$
$$h(t-T) + p^- F_{G,T}(Y^*(t)) - p^+\left(1 - F_{t,T}(Y^*(t))\right) = 0, \text{ respectively.}$$

Proof: Given (29) and (30), we can formalize the case of $\psi(t,\xi) = 0$ in optimality conditions (14). To determine the control in such a case over an interval, we differentiate this condition over this interval. Taking into account (29), we find for $t \geq G$, $\dot{\psi}(t,\xi) = h + (p^- + p^+)\partial F_{t,T}(Y(t,\xi))/\partial t = 0$.

Denote the $u(t,\xi)$ that satisfies $\partial F_{t,T}(Y(t,\xi))/\partial t = -h/(p^- + p^+)$ as $u^*(t)$ and the $X(t,\xi)$ that satisfies $h(t-T) - \left(p^+ - (p^- + p^+)F_{t,T}(Y(t,\xi))\right) = 0$ as $X^*(t)$. Since $(\partial(p^- + p^+)F_{t,T}(Y(t,\xi))/\partial X(t,\xi)) =$

$-((p^- + p^+)/U)(\partial F_{t,T}(Y(t,\xi))/\partial Y(t,\xi)) \leq 0$, we can express optimality conditions (14) for $t \geq G$ as shown in

$$u(t,\xi)=\begin{cases} U, & \text{if } X(t,\xi) < X^*(t) \text{ and } \alpha(t,\xi)=1 \\ u^*(t), & \text{if } X(t,\xi)=X^*(t) \text{ and } \alpha(t,\xi)=1 \\ 0, & \text{if either } X(t,\xi) > X^*(t) \text{ or } \alpha(t,\xi)=0, \text{ or both} \end{cases} \quad (31)$$

as stated in the theorem. Applying the same arguments the optimal feedback policy is obtained for $t < G$. ■

Note, that Theorem 1 assumes that the unit inventory holding cost over the entire horizon is larger than the unit shortage cost, that is, $hT > p^-$. If the unit holding cost is smaller, that is, $hT \leq p^-$, then $t \geq 0 \geq T - (p^-/h)$ holds and thus the first policy determined in Theorem 1 is the only policy which is always optimal.

Extension to Random Demand

The effect of the random demand is straightforward. Because breakdowns do not depend on demands, the only change we need in (29) and (30) is an additional integral over demand D as follows:

$$\begin{aligned}\psi(t,\xi) &= h(t-T) \\ &\quad - \int_0^\infty \left(p^+ - (p^- + p^+)F_{t,T}(Y(t,\xi))\right) \\ &\quad \times f_d(D)dD \text{ if } t \geq G \qquad (32)\\ \psi(t,\xi) &= h(t-T) \\ &\quad + \int_0^\infty \left[p^- F_{G,T}(Y(t,\xi)) - p^+(1 - F_{t,T}(Y(t,\xi))\right] \\ &\quad \times f_d(D)dD \text{ if } t < G \qquad (33)\end{aligned}$$

respectively. Therefore, the optimal feedback policy (31) remains the same, but $u^*(t)$ and $X^*(t)$ satisfy

$$\begin{aligned}&\int_0^\infty \frac{\partial F_{t,T}(Y^*(t))}{\partial t} f_d(D)dD \\ &\quad = -\frac{h}{p^- + p^+} \\ &h(t-T) - \int_0^{+\infty} \left(p^+ - (p^- + p^+)F_{t,T}(Y^*(t))\right) f_d(D)dD \\ &\quad = 0 \text{ for } t \geq G. \qquad (34)\end{aligned}$$

Example: Suppose both demand and machine state are characterized by uniform distributions:

$$f_{t,T}(A) = \begin{cases} 1/(T-t), & \text{if } 0 \leq A \leq T-t \\ 0, & \text{otherwise} \end{cases}$$

$$f_d(D) = \begin{cases} (1/M), & \text{if } 0 \leq D \leq M \\ 0, & \text{otherwise} \end{cases}$$

and $hT \leq p$. Then if $t \geq G = T - (p^-/h)$, $u^*(t) = (p^+/(p^+ + p^-))U + (2h/(p^+ + p^-))U(T-t)$ and $X^*(t) = (M/2) - ((p^+ + h(T-t))/(p^+ + p^-))U(T-t)$.

This implies that even under a simple uniform distribution, the optimal threshold $X^*(t)$ is a nonlinear function of time. It increases monotonically with a decreasing rate, $\ddot{X}^*(t) = -(2h/(p^+ + p^-))U$, (in a concave manner) and attains the expected demand, $E[d] = M/2$ at $t = T$. Furthermore, if $UT > M/2$, i.e., the system is capable of meeting the average demand, no production is needed at the beginning since $X^*(0) < 0$. Once $X^*(t)$ becomes equal to the current inventory level $X(t,\xi)$, the optimal control switches. Specifically, each time the machine is down and thus $X(t,\xi) < X^*(t)$, the maximum production rate will be used to restore inventories as fast as possible upon the machine repair. When $X(t,\xi)$ reaches $X^*(t)$, the maximum rate will be switched to the intermediate production rate $u^*(t)$, and the inventory level will be maintained at $X^*(t)$ until the next breakdown.

TABLE I
PERFORMANCE OF THE SUGGESTED METHODOLOGY VERSUS THE OPTIMAL SOLUTION

	Time-dependent distribution	Time-independent distribution
Under capacity situation	Relative Gap, %	
Breakdown level-15%	1.308748	0.095177
Breakdown level-10%	1.377124	0.073685
Breakdown level- 5%	0.504256	0.055248
Over capacity situation	Relative Gap, %	
Breakdown level-15%	0.000106	8.42E-05
Breakdown level-10%	0.000112	0.001056
Breakdown level- 5%	0	3.53E-05

VI. Simulation Analysis

In this section, we use simulation to compare the suggested decomposition approach and the optimal solution. To do so, we first rewrite (10) as the following:

$$\delta J = \varepsilon\alpha(t,\xi)\delta u h(T-t) + \frac{d}{dX(T,\xi)} \underset{R(t,\xi)}{E}\left[p^- \max\{0, D - X(T,\xi)\} + p^+ \max\{0, X(T,\xi) - D\}\right]\delta X(T,\xi). \quad (35)$$

Using (9), we thus obtain an equivalent expression for (11):

$$\psi(t,\xi) = h(t-T) + \underset{R(t,\xi)}{E}[\gamma(X(T,\xi) - D)] \quad (36)$$

where

$$\gamma(X(T,\xi) - D) = \begin{cases} -p^+, & \text{if } X(T,\xi) > D \\ p^-, & \text{if } X(T,\xi) < D \\ p \in [-p^+, p^-], & \text{if } X(T,\xi) = D. \end{cases} \quad (37)$$

Substituting (36) and (37) into $\psi(t,\xi) = 0$ from general stochastic optimality condition (14) we have the following equation for $X^*(t)$:

$$\begin{aligned}h(T-t) &= \underset{R(t,\xi)}{E}[\gamma(X(T,\xi) - D)] \\ &= \underset{R(t,\xi)}{E}\left[\gamma\left(X^*(t) + \int_t^T \alpha(\tau,\xi)u(\tau,\xi)d\tau - D\right)\right] \quad (38)\end{aligned}$$

where $u(t,\xi)$ is determined by (31). Now, instead of analysis of the expectation in (38) as suggested in this note, we use simulation to calculate it.

Given a probability distribution function, at each time point t we now can simulate the machine state for a very short interval ε to study the effect of optimal threshold (38) compared to that of the decomposition approach (Theorem 1) on the expected cost. For example, if $t \geq T -$

TABLE II
THRESHOLDS OF THE SUGGESTED METHODOLOGY VERSUS THE OPTIMAL SOLUTION

t	0	1	2	3	4	5	6	7	8	9	10
Suggested methodology											
$X^*(t)$	-15.0	-10.5	-5.5	-1.0	3.5	8.5	13.0	17.5	22.0	26.5	30.0
Optimal solution											
$X^*(t)$	-15.2	-10.9	-5.6	-0.7	3.5	8.2	13.9	17.8	21.6	26.45	30.0

(p^-/h), the threshold $X^*(t)$ at t and $t+\varepsilon$ determined by Theorem 1 are

$$h(t-T) + p^- F_{T-(p^-/h),T}\left(\frac{D-X^*(t)}{U}\right) - p^+\left(1 - F_{t,T}\left(\frac{D-X^*(t)}{U}\right)\right) = 0$$

$$h(t+\varepsilon-T) + p^- F_{T-(p^-/h),T}\left(\frac{D-X^*(t+\varepsilon)}{U}\right) - p^+\left(1 - F_{t,T}\left(\frac{D-X^*(t+\varepsilon)}{U}\right)\right) = 0$$

and the production rate if $X(t) = X^*(t)$ is $u^*(t) = (X^*(t+\varepsilon) - X^*(t))/\varepsilon$.

The procedure for constructing optimal thresholds is as follows: select time step ε, threshold accuracy θ, set $t = T-\varepsilon$ and $X^*(T) = D$, i) simulate breakdowns starting from t for each $X^i(t)+i\theta, i = 0, 1, \ldots$, and, $X^0(t) = 0$ until i^* is found so that (38) is met for $X^*(t) = X^{i*}(t) + i^*\theta$; ii) set $t = t-\varepsilon$, if $t > 0$ go to i), otherwise stop, all thresholds have been found. The procedure is backward as to find threshold $X^*(t')$ from (38), we need controls and, thus, thresholds over all $t > t'$. Once these true optimal values $X^*(t)$ for the entire planning horizon are found, we proceed with the standard forward simulation using the optimal thresholds and calculate the average objective function (5) for interval $[t, T]$. This average is then compared with the one obtained by using the decomposition method.

Two types of experiments were conducted, one assumes that the machine up and down times follow uniform distributions characterized by $b_{\rm up}$, $a_{\rm up}$ and $b_{\rm down}$, $a_{\rm down}$, the other, the time-independent Bernoulli distribution with the machine being up at each ε with probability p. In the former case probability density function $f_{t,T}(A)$ was calculated numerically, while in the latter, it is binomial over each interval $[t, T]$. All experiments were divided into two groups characterized by over-capacity $(pUT > D, ((b_{\rm up} - a_{\rm up})/(b_{\rm down} - a_{\rm up}))UT > D)$ and under-capacity, respectively. The results of more than a hundred examples show that the relative difference between the average cost of the optimal solution and that of the decomposition method is no more than 0.1% for the analytical (binomial) distribution and no more than 2.4% for the numerically obtained time-dependent distribution. In addition, in all experiments, the time needed to compute thresholds was less than half an hour for the numerical distribution (a few seconds for the analytical one), while it was more than forty seven hours computation to reach the optimality. Table I presents the results for the analytical and the numerical distributions under both under-capacity $(D = 30)$ and over-capacity $(D = 60)$ conditions with $T = 10$, $X_0 = 0$, $U = 5$, $h = 2$, $p^+ = 1$, and $p^- = 30$, $\varepsilon = 0.1$ and $\theta = 0.01$. Each of the examples was simulated for 1 000 000 runs, for three breakdown levels ($(1-p)100\%$ in analytical case and $(1 - ((b_{\rm up} - a_{\rm up})/(b_{\rm down} - a_{\rm up})))100\%$ in numerical case) of 5%, 10%, and 15%.

Table II presents optimal thresholds versus those obtained with the suggested methodology for the example of Table I (time-independent distribution with under capacity, $p = 0.9$).

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