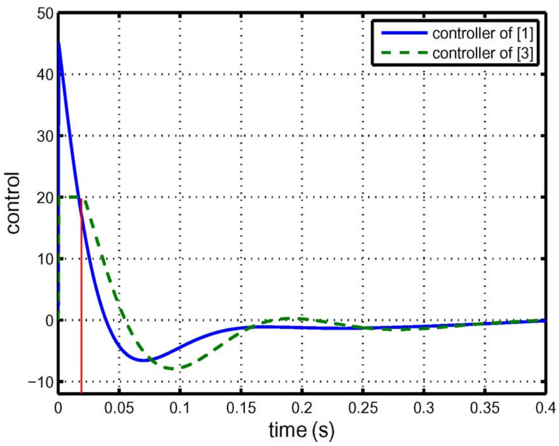

Fig. 5. Control comparison of the controllers of [1] and [3].

VII. Conclusion

We have shown that the transient response of the controller of [1] recovers the response of a high-gain feedback controller without internal model. On the other hand, the transient response of the controller of [3] recovers the response of a sliding mode controller without internal model. These properties show advantage of the designs of [1] and [3] over other designs for the stabilization of the augmented system of the plant and the internal model. Because of the connection between sliding mode and high-gain feedback controllers, the designs of [1] and [3] are indeed close to each other. The difference between them boils down to the difference between high-gain feedback and sliding mode control, where high-gain feedback drives the trajectories towards a slow manifold faster than a sliding mode control driving the trajectories towards a sliding manifold. This happens at the expense of a larger control signal for the high-gain feedback controller during the transient period.

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Continuous-Time Replenishment Under Intermittent Observability

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***Abstract*—In this technical note we study continuous-time stochastic control of a dynamic production and replenishment system characterized by bounded control and an additive type of uncertainty. The study is motivated by problems arising in supply chains involving periodic exchange of information between a manufacturing system (supplier) and a customer (retailer). As a result, the inventories are only observed periodically while the replenishment is possible at any point of time. We identify replenishment policies for different operational conditions and show that, even for one-product-type system, the consideration of random demand over multiple update periods leads to a non-intuitive, and nontrivial, optimal production control.**

***Index Terms*—Continuous replenishment, periodic updates, stochastic inventory control.**

I. Introduction

Classical multi-period (discrete-time) stochastic inventory problems are usually treated using the recursive dynamic programming approach (see, for example, Zipkin [10] for a variety of this type of models). Classical stochastic inventory problems with continuous inventory updates are commonly treated using the continuous-time dynamic programming approach (Hamiltonian-Jacobi-Bellman equation) (see, for example, the pioneering work of Kimemia [6], Kimemia and Gershwin [7], Ghosh *et al.* [3], and Akella and Kumar [1]) and the maximum principle, if no updates are available during the planning horizon (e.g., see Khmelnitsky and Caramanis, [5]; Kogan *et al.* [8], Kogan and Lou, [9]).

This work considers a continuous stochastic control (inventory replenishment) problem under periodic updates and thus deals with the challenge of integrating the above streams of research. The problem is due to a relatively new approach to the allocation of responsibility in the replenishment process, which is referred to as Vendor Managed Inventory (VMI). As opposed to traditional orders, where the customer makes the replenishment decision, the VMI approach implies that the supplier makes this decision on the customer's behalf (Harrison and van Hoek, [4]; Disney and Towill, [2]). The decision is based on the information, which is transferred between the parties periodically. Specifically, the updates on the inventory level of the retailer are delivered periodically to the manufacturer, while the manufacturer who handles the retailer's inventories can replenish them at any point of time. The approach, which we suggest to study such a system, is based on: (i) the recursive discrete-time dynamic programming for over stage global optimization upon updates and (ii) the continuous-time maximum principle for optimizing the Bellman (cost-to-go) function in between the updates, i.e., at each separate period and thereby stage of dynamic programming. As a result, we derive an optimal solution and show that the optimal control is piece-wise constant and has at most two switching points at each period. We also discuss the affect of sales lost at the end of each period on the optimal solution.

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II. Statement of the Problem

Consider a manufacturer who produces and supplies a single-product-type to a retailer. Since the demand for the product is random, the retailer periodically provides the manufacturer with updated position of its inventory. Let n be the index of review periods, and there will be N such periods, $n = 1, \ldots, N$, of length τ. Then period n is determined by time t such that $(n-1)\tau < t \le n\tau$, for $n = 1, \ldots, N$. Let the supplier choose a production plan which is a replenishment policy with respect to the retailer. The replenishment rate (in terms of the retailer) or the production rate (in terms of the manufacturer), $u(t)$, is bounded and controllable, i.e.,

$$0 \le u(t) \le U. \quad (1)$$

Given fluid material flow over a fixed production horizon, $[0, T]$, the retailer's inventory process $X(t)$ is described by the following dynamics:

$$X(t) = X^{n-1} + \int_{(n-1)\tau}^{t} (u(s) - D_n)\, ds \quad (2)$$

for $(n-1)\tau \le t < n\tau$, $n = 1, 2, \ldots, N$ or, if sales of period n are lost once the period has been completed, i.e., backlogs are limited to the same period, then

$$X(t) = \max\{X^{n-1}, 0\} + \int_{(n-1)\tau}^{t} (u(s) - D_n)\, ds \quad (3)$$

for $(n-1)\tau \le t < n\tau$, $n = 1, 2, \ldots, N$.

In (2) and (3), X^n is the inventory level at $t = n\tau$ and D_n is the realization of a random demand rate, d_n, at period n. We denote by $f_n(D_n)$ and $F_n(.)$ the density and cumulative distribution functions of the demand respectively. Note that, since no new information will become available during a period, n, (X^k, $k = n, \ldots, N$ are unknown at period n) the determination of how much to produce (replenish) and when to produce must be made based only on the last inventory update (review), X^{n-1}, and before production of period n commences.

The objective is to determine the replenishment rule $\{u(t)|X^{n-1} : (n-1)\tau < t \le n\tau\}$ for each period $n = 1, \ldots, N$ over the entire production horizon T in order to minimize the expected inventory cost

$$J(u, X^0) = E\left[\int_0^T g(X(t))\, dt\right] \quad (4)$$

where $g(\cdot)$ is a piecewise linear cost function, $g(X(t)) = c^+X^+(t) + c^-X^-(t)$, and c^+, c^- are the nonnegative inventory surplus and backlog cost coefficients, respectively, $X^+(t) = \max\{0, X(t)\}$, and $X^-(t) = \max\{0, -X(t)\}$.

III. Over-Stage Optimization Approach

We let the production policy during period n be $u_n(\cdot)$, i.e., $u_n(\cdot) = u_n(t)$ for $(n-1)\tau < t \le n\tau$, $n = 1, 2, \ldots, N$ and introduce a new notation, $\mathbf{u}^n = [u_n(\cdot), \ldots, u_N(\cdot)]$. We next present the function

$$J_n(\mathbf{u}^n, X^{n-1}) = E\left[\sum_{i=n}^{N} \int_{(i-1)\tau}^{i\tau} g(X(t))\, dt\right] \quad (5)$$

which is evidently equivalent to the objective function (4), when $n = 1$. Then the *Bellman* (cost-to-go) function is

$$B_n(X^{n-1}) = \min_{\mathbf{u}^n} J_n(\mathbf{u}^n, X^{n-1}) \quad n = 1, \ldots, N. \quad (6)$$

Consequently, introducing for convenience, $G(u_n(.), X^{n-1}) = \int_{(n-1)\tau}^{n\tau} g(X(t))dt$, the principle of optimality straightforwardly results in the following recursive dynamic programming equations:

$$B_n(X^{n-1}) = \min_{u_n} \left\{E_n\left[G\left(u_n(.), X^{n-1}\right) + B_{n+1}(X^n)\right]\right\},$$
$$n = 1, \ldots, N, \quad B_{N+1}(X^N) = 0. \quad (7)$$

The index in the expectation E_n implies that the expectation is taken at period n. In the next sections we show that at each stage n of the recursive dynamic programming we solve a canonical optimal control problem to minimize the cost-to-go function with controls $u_n(\cdot)$.

IV. The In-Stage Optimization Approach

Assume first that there are two periods left to go, i.e., that we are at period $N-1$. To proceed, we employ b_n that satisfies $F_n(b_n) = c^-/(c^+ + c^-)$. To facilitate the presentation, we assume that

$$0 < b_n < U, \quad 1 \le n \le N. \quad (8)$$

Parameter b_n, as will be shown below, is an optimal replenishment level. Therefore, assumption (8) implies that the manufacturer has a sufficient capacity, U, to provide optimal supplies. In addition, we assume that function $f_n(D_n)$ is positive (i.e. does not vanish) at the interval $(\inf d_n, \sup d_n)$, where $\inf d_n = \inf\{D|f_n(D) > 0\}$ and $\sup d_n = \sup\{D|f_n(D) > 0\}$. Let $t \in ((n-1)\tau, n\tau]$, $1 \le n \le N$, and denote

$$Y(t) = X^{n-1} + \int_{(n-1)\tau}^{t} u_n(s)ds. \quad (9)$$

Then from (5) we obtain (10), shown at the bottom of the next page. Since, $X^{N-1} = Y((N-1)\tau) - D_{N-1}\tau$ or, for the case of lost sales, $X^{N-1} = \max\{Y((N-1)\tau) - D_{N-1}\tau, 0\}$, we have $X^{N-1} = X^{N-1}(Y((N-1)\tau), D_{N-1})$. That is, $E_{N-1}[B_N(X^{N-1})]$ depends on $Y((N-1)\tau)$. Denote $\varphi_2(Y((N-1)\tau)) = E_{N-1}[B_N(X^{N-1})]$. Then employing our notation for φ_2 we find from (7)

$$B_{N-1}(X^{N-2}) = \min_{u_{N-1}} E_{N-1}\left[G\left(u_{N-1}(.), X^{N-2}\right)\right]$$
$$+\varphi_2\left(Y\left((N-1)\tau\right)\right). \quad (11)$$

Namely (12), shown at the bottom of the next page. This implies that at step $n = N-1$ of the recursive dynamic programming, we solve an optimal control problem (12), (9) and (1) to minimize the cost-to-go function with control $u_{N-1}(\cdot)$. To analyze the problem, we construct the Hamiltonian

$$\begin{cases} H\left(Y(t), \psi(t), u_{N-1}(t)\right) = \psi(t)u_{N-1}(t) \\ \quad - \int_{-\infty}^{\frac{Y(t)}{t-(N-2)\tau}} c^+\left(Y(t) - D_{N-1}\left[t-(N-2)\tau\right]\right) \\ \quad \quad \times f_{N-1}(D_{N-1})dD_{N-1} \\ \quad + \int_{\frac{Y(t)}{t-(N-2)\tau}}^{\infty} c^-\left(Y(t) - D_{N-1}\left[t-(N-2)\tau\right]\right) \\ \quad \quad \times f_{N-1}(D_{N-1})dD_{N-1} \end{cases} \quad (13)$$

and the co-state differential equation for $(N-2)\tau < t \le (N-1)\tau$

$$\begin{cases} \dot{\psi}(t) = -\frac{\partial H\left(Y(t), \psi(t), u_{N-1}(t)\right)}{\partial Y(t)} \\ \quad = (c^+ + c^-)F_{N-1}\left(\frac{Y(t)}{t-(N-2)\tau}\right) - c^- \\ \psi\left((N-1)\tau\right) = -\frac{\partial \varphi_2(Y(N-1)\tau))}{\partial Y(N-1)\tau)}. \end{cases} \quad (14)$$

Maximizing Hamiltonian (13) we readily observe that, at period $N-1$, the optimal production rate is given by

$$u_{N-1}(t) = \begin{cases} U, & \psi(t) > 0 \\ b_{N-1}, & \psi(t) = 0 \\ 0, & \psi(t) < 0 \end{cases} \tag{15}$$

for $(N-2)\tau < t \le (N-1)\tau$, where the co-state variable $\psi(t)$ is defined by (14) for $(N-2)\tau \le t \le (N-1)\tau$.

In what follows we assume that

$$\frac{\partial^2 \varphi_2\left(Y\left((N-1)\tau\right)\right)}{\partial Y^2\left((N-1)\tau\right)} \ge 0. \tag{16}$$

In Section VI we show that the cost-to-go function $B_N(X^{N-1})$ is convex and therefore the function φ_2 is convex, which ensures (16).

V. Optimal Policies

Following lemmas describe all possible optimal solutions at period $N-1$ when the initial inventory at this period is non-negative. To simplify the presentation in this section, we omit the index at φ_2.

Lemma 1: Assume that $X^{N-2} = 0$ and $-(\partial\varphi(Y((N-1)\tau))/\partial Y((N-1)\tau))|_{Y((N-1)\tau)=U\tau} \ge 0$. If

$$-\left.\frac{\partial \varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\right|_{Y((N-1)\tau)=U\tau} - \int_{(N-2)\tau}^{(N-1)\tau} \left[(c^+ + c^-)F_{N-1}(U) - c^-\right] ds \le 0 \tag{17}$$

then the optimal solution is

$$u_{N-1}(t) = U, \quad t \in \left((N-2)\tau, (N-1)\tau\right]. \tag{18}$$

Otherwise, the solution is

$$u_{N-1}(t) = \begin{cases} b_{N-1}, & (N-2)\tau < t \le t_1 \\ U, & t_1 < t \le (N-1)\tau \end{cases} \tag{19}$$

where t_1 satisfies

$$\begin{cases} -\left.\frac{\partial\varphi(Y((N-1)\tau))}{\partial Y((N-1)\tau)}\right|_{Y((N-1)\tau)=b_{N-1}(t_1-(N-2)\tau)+U((N-1)\tau-t_1)} \\ \quad - \int_{t_1}^{(N-1)\tau} \left[(c^+ + c^-)F_{N-1}\right. \\ \qquad \left. \times \left(\frac{b_{N-1}(t_1-(N-2)\tau)+U(t-t_1)}{t-(N-2)\tau}\right) - c^-\right] dt = 0. \end{cases} \tag{20}$$

$$\begin{cases} J_{N-1}(\mathbf{u}^{N-1}, X^{N-2}) = E_{N-1}\left[G(u_{N-1}, X^{N-2})\right] + E_{N-1}\left[E_N\left[G(u_N, X^{N-1})\right]\right] \\ = \int_{(N-2)\tau}^{(N-1)\tau} dt \left[\int_{-\infty}^{\frac{Y(t)}{t-(N-2)\tau}} c^+\left(Y(t) - D_{N-1}\left(t-(N-2)\tau\right)\right) f_{N-1}(D_{N-1}) dD_{N-1} \right. \\ \qquad \left. - \int_{\frac{Y(t)}{t-(N-2)\tau}}^{\infty} c^-\left(Y(t) - D_{N-1}\left(t-(N-2)\tau\right)\right) f_{N-1}(D_{N-1}) dD_{N-1} \right] \\ \quad + \int_{-\infty}^{\infty} f_{N-1}(D_{N-1}) dD_{N-1} \int_{(N-1)\tau}^{N\tau} dt \left[\int_{-\infty}^{\frac{Y(t)}{t-(N-1)\tau}} c^+\left(Y(t) - D_N\left(t-(N-1)\tau\right)\right) f_N(D_N) dD_N \right. \\ \qquad \left. - \int_{\frac{Y(t)}{t-(N-1)\tau}}^{\infty} c^-\left(Y(t) - D_N\left(t-(N-1)\tau\right)\right) f_N(D_N) dD_N \right] \\ = \int_{(N-2)\tau}^{(N-1)\tau} dt \left[\int_{-\infty}^{\frac{Y(t)}{t-(N-2)\tau}} c^+\left(Y(t) - D_{N-1}\left(t-(N-2)\tau\right)\right) f_{N-1}(D_{N-1}) dD_{N-1} \right. \\ \qquad \left. - \int_{\frac{Y(t)}{t-(N-2)\tau}}^{\infty} c^-\left(Y(t) - D_{N-1}\left(t-(N-2)\tau\right)\right) f_{N-1}(D_{N-1}) dD_{N-1} \right] \\ \quad + E_{N-1}\left[J_N(u_N, X^{N-1})\right] \end{cases} \tag{10}$$

$$\begin{cases} B_{N-1}(X^{N-2}) = \min_{u_{N-1}} \left\{ \int_{(N-2)\tau}^{(N-1)\tau} dt \left[\int_{-\infty}^{\frac{Y(t)}{t-(N-2)\tau}} c^+\left(Y(t) - D_{N-1}\left[t-(N-2)\tau\right]\right) f_{N-1}(D_{N-1}) dD_{N-1} \right. \right. \\ \qquad \left. \left. - \int_{\frac{Y(t)}{t-(N-2)\tau}}^{\infty} c^-\left(Y(t) - D_{N-1}\left[t-(N-2)\tau\right]\right) f_{N-1}(D_{N-1}) dD_{N-1} \right] \right\} \\ \quad + \varphi_2\left(Y\left((N-1)\tau\right)\right) \end{cases} \tag{12}$$

Lemma 2: Assume that $X^{N-2} = 0$. If

$$-\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=U\tau} < 0 \leq -\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=b_{N-1}\tau}$$

then the optimal solution is (19). If

$$-\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=b_{N-1}\tau} < 0 < -\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=0}$$

then

$$u_{N-1}(t) = \begin{cases} b_{N-1}, & (N-2)\tau < t \leq t_2 \\ 0, & t_2 < t \leq (N-1)\tau \end{cases} \tag{21}$$

where t_2 satisfies

$$\begin{cases} -\frac{\partial\varphi(Y((N-1)\tau)))}{\partial Y((N-1)\tau))}\Big|_{Y(\tau)=b_{N-1}(t_2-(N-2)\tau)} \\ \quad -\int\limits_{t_2}^{(N-1)\tau}\Big[(c^+ + c^-)F_{N-1} \\ \qquad \times\left(\frac{b_{N-1}(t_2-(N-2)\tau)}{s-(N-2)\tau}\right) - c^-\Big]\,ds = 0. \end{cases}$$

Lemma 3: Assume that $X^{N-2} = 0$ and $-(\partial\varphi(Y((N-1)\tau))/\partial Y((N-1)\tau))|_{Y((N-1)\tau)=0} \leq 0$. If

$$-\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=0} - \int\limits_{(N-2)\tau}^{(N-1)\tau}\left[(c^+ + c^-)F_{N-1}(0) - c^-\right]ds \leq 0$$

then the optimal solution is

$$u_{N-1}(t) = 0, \quad t \in ((N-2)\tau, (N-1)\tau]. \tag{22}$$

Otherwise, the solution is (21).

Lemma 4: Assume that $X^{N-2} \geq b_{N-1}\tau$ and $-(\partial\varphi(Y((N-1)\tau))/\partial Y((N-1)\tau))|_{Y((N-1)\tau)=X^{N-2}+U\tau} \geq 0$. If

$$\begin{cases} -\frac{\partial\varphi(Y((N-1)\tau))}{\partial Y((N-1)\tau)}\Big|_{Y((N-1)\tau)=X^{N-2}+U\tau} \\ \quad -\int\limits_{(N-2)\tau}^{(N-1)\tau}\Big[(c^+ + c^-)F_{N-1} \\ \qquad \times\left(\frac{X^{N-2}+U(s-(N-2)\tau)}{s-(N-2)\tau}\right) - c^-\Big]\,ds \geq 0 \end{cases} \tag{23}$$

then the optimal solution is (18). Otherwise, the solution is

$$u_{N-1}(t) = \begin{cases} 0, & (N-2)\tau < t \leq t_3 \\ U, & t_3 < t \leq (N-1)\tau \end{cases} \tag{24}$$

where t_3 satisfies

$$\begin{cases} -\frac{\partial\varphi(Y((N-1)\tau))}{\partial Y((N-1)\tau)}\Big|_{Y((N-1)\tau)=X^{N-2}+U((N-1)\tau-t_3)} \\ \quad -\int\limits_{t_3}^{(N-1)\tau}\Big[(c^+ + c^-)F_{N-1}\Big(\frac{X^{N-2}+U(s-t_3)}{s-(N-2)\tau}\Big) - c^-\Big]ds = 0. \end{cases}$$

Lemma 5: Assume that $X^{N-2} \geq b_{N-1}\tau$. If

$$-\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=X^{N-2}+U\tau} < 0 < -\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=X^{N-2}}$$

then the optimal solution is (24). Otherwise, if

$$-\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=X^{N-2}} \leq 0$$

the optimal solution is (22).

Lemma 6: Assume that $0 < X^{N-2} < b_{N-1}\tau$ and

$$-\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=X^{N-2}+U\tau} \geq 0.$$

If

$$\begin{cases} -\frac{\partial\varphi(Y((N-1)\tau))}{\partial Y((N-1)\tau)}\Big|_{Y((N-1)\tau)=X^{N-2}+U\tau} \\ \quad -\int\limits_{(N-2)\tau}^{(N-1)\tau}\Big[(c^+ + c^-)F_{N-1} \\ \qquad \times\left(\frac{X^{N-2}+U(s-(N-2)\tau)}{s-(N-2)\tau}\right) - c^-\Big]\,ds \geq 0 \end{cases}$$

then the optimal solution is (18).

If you have the equation shown at the bottom of the page, then the solution is (24). Otherwise, if you have the first equation shown at the bottom of the next page, then

$$u_{N-1}(t) = \begin{cases} 0, & (N-2)\tau < t < (N-2)\tau + \frac{X^{N-2}}{b_{N-1}} \\ b_{N-1}, & (N-2)\tau + \frac{X^{N-2}}{b_{N-1}} \leq t < t_4 \\ U, & t_4 \leq t \leq (N-1)\tau \end{cases} \tag{25}$$

$$\begin{cases} -\frac{\partial\varphi(Y((N-1)\tau))}{\partial Y((N-1)\tau)}\Big|_{Y((N-1)\tau)=X^{N-2}+U\tau} \\ \quad -\int\limits_{(N-2)\tau}^{(N-1)\tau}\Big[(c^+ + c^-)F_{N-1}\left(\frac{X^{N-2}+U(s-(N-2)\tau)}{s-(N-2)\tau}\right) - c^-\Big]\,ds < 0 \\ \leq -\frac{\partial\varphi(Y((N-1)\tau))}{\partial Y((N-1)\tau)}\Big|_{Y((N-1)\tau)=X^{N-2}+U\left(\tau-\frac{X^{N-2}}{b_{N-1}}\right)} \\ \quad -\int\limits_{(N-2)\tau+\frac{X^{N-2}}{b_{N-1}}}^{(N-1)\tau}\left[(c^+ + c^-)F_{N-1}\left(\frac{X^{N-2}+U\left(s-\left((N-2)\tau+\frac{X^{N-2}}{b_{N-1}}\right)\right)}{s-(N-2)\tau}\right) - c^-\right]ds \end{cases}$$

where t_4 satisfies

$$\begin{cases} -\frac{\partial\varphi(Y((N-1)\tau))}{\partial Y((N-1)\tau)}\Big|_{Y((N-1)\tau)=b_{N-1}(t_4-(N-2)\tau)+U((N-1)\tau-t_4)} \\ \quad -\int\limits_{t_4}^{(N-1)\tau}\Big[(c^+ + c^-)F_{N-1} \\ \qquad \left(\frac{b_{N-1}(t_4-(N-2)\tau)+U(s-t_4)}{s-(N-2)\tau}\right) - c^-\Big]\,ds = 0. \end{cases}$$

Lemma 7: Assume that $0 < X^{N-2} < b_{N-1}\tau$ and

$$-\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=X^{N-2}+U\tau} < 0 \le -\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=X^{N-2}+U\left(\tau-\frac{X^{N-2}}{b_{n-1}}\right)}$$

If you have (26), shown at the bottom of the page, then the optimal solution is (24). Otherwise, if (26) is not met, then the optimal solution is (25).

Lemma 8: Assume that $0 < X^{N-2} < b_{N-1}\tau$. If

$$-\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=X^{N-2}+U\left(\tau-\frac{X^{N-2}}{b_{N-1}}\right)} < 0 \le -\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=b_{N-1}\tau}$$

then the optimal solution is (25). Otherwise, if

$$-\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=b_{N-1}\tau} < 0 \le -\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=X^{N-2}}$$

then the optimal solution is

$$u_{N-1}(t) = \begin{cases} 0, & (N-2)\tau < t < (N-2)\tau + \frac{X^{N-2}}{b_{N-1}} \\ b_{N-1}, & (N-2)\tau + \frac{X^{N-2}}{b_{N-1}} \le t \le t_5 \\ 0, & t_5 < t \le (N-1)\tau \end{cases} \tag{27}$$

where t_5 satisfies

$$\begin{cases} -\frac{\partial\varphi(Y((N-1)\tau))}{\partial Y((N-1)\tau)}\Big|_{Y((N-1)\tau)=b_{N-2}(t_5-(N-2)\tau)} \\ \quad -\int\limits_{t_5}^{(N-1)\tau}\Big[(c^+ + c^-)F_{N-1}\left(\frac{b_{N-1}(t_5-(N-2)\tau)}{s-(N-2)\tau}\right) - c^-\Big]ds \\ \quad = 0. \end{cases} \tag{28}$$

Lemma 9: Assume that $0 < X^{N-2} < b_{N-1}\tau$ and $-(\partial\varphi(Y((N-1)\tau))/\partial Y((N-1)\tau))|_{Y((N-1)\tau)=X^{N-2}} < 0$. If

$$-\frac{\partial\varphi\left(Y\left((N-1)\tau\right)\right)}{\partial Y\left((N-1)\tau\right)}\bigg|_{Y((N-1)\tau)=X^{N-2}} - \int\limits_{(N-2)\tau+\frac{X^{N-2}}{b_{N-1}}}^{(N-1)\tau}\left[(c^+ + c^-)F_1\left(\frac{X^{N-2}}{s-(N-2)\tau}\right) - c^-\right]ds \le 0 \tag{29}$$

then the optimal solution is (22). Otherwise, the solution is (27).

Lemmas 1–9 show that an optimal solution is piecewise constant and includes at most three production regimes. Specifically, the production can be switched between the maximum rate, intermediate rate and no production at all.

VI. Generalization

The subsequent lemma implies that all the optimal policies determined in Lemmas 1–9 are valid for any review period. Specifically, we next show that the dependence of φ_2 on $Y((N-1)\tau)$ see ((11)) holds for any period, i.e., φ_n depends on $Y([N-(n-1)]\tau)$. The proof is by induction with n denoting the number of periods *left to go*, rather than the current period. That is, the current period is $N-n+1$.

Lemma 10: Assume there are n periods left to go $(2 \le n \le N)$, then

$$\varphi_n = \varphi_n\left(Y\left([N-n+1]\tau\right)\right). \tag{30}$$

According to Lemma 10, the dynamic programming (11) takes the following form:

$$\begin{cases} B_{N-n+1}(X^{N-n}) \\ \quad = \min\limits_{u_{N-n+1}} E_{N-n+1}\left[G\left(u_{N-n+1}(.), X^{N-n}\right)\right] \\ \quad +\varphi_n\left(Y\left((N-n+1)\tau\right)\right). \end{cases} \tag{31}$$

$$\begin{cases} -\frac{\partial\varphi(Y((N-1)\tau))}{\partial Y((N-1)\tau)}\Big|_{Y((N-1)\tau)=X^{N-2}+U\left(\tau-\frac{X^{N-2}}{b_{N-1}}\right)} \\ \quad -\int\limits_{(N-2)\tau+\frac{X^{N-2}}{b_{N-1}}}^{(N-1)\tau}\left[(c^+ + c^-)F_{N-1}\left(\frac{X^{N-2}+U\left(s-\left((N-2)\tau+\frac{X^{N-2}}{b_{N-1}}\right)\right)}{s-(N-2)\tau}\right) - c^-\right]ds \\ \quad < 0 \end{cases}$$

$$\begin{cases} -\frac{\partial\varphi(Y((N-1)\tau))}{\partial Y((N-1)\tau)}\Big|_{Y((N-1)\tau)=X^{N-2}+U\left(\tau-\frac{X^{N-2}}{b}\right)} \\ \quad -\int\limits_{X^{N-2}+U\left(\tau-\frac{X^{N-2}}{b}\right)}^{(N-1)\tau}\left[(c^+ + c^-)F_1\left(\frac{X^{N-2}+U\left(s-\left((N-2)\tau+\frac{X^{N-2}}{b}\right)\right)}{s-(N-2)\tau}\right) - c^-\right]ds \\ \quad \ge 0 \end{cases} \tag{26}$$

This implies that at step $N-n+1$ of the recursive dynamic programming, we solve an optimal control problem to minimize the cost-to-go function with controls $u_{N-n+1}(\cdot)$.

We next index the periods in the natural order, i.e., index n implies the current period, $n = 1, \ldots, N$, and prove the convexity of the Bellman function (6).

Theorem 1: The function $B_n(X^{n-1})$ is convex in X^{n-1} for every $1 \leq n \leq N$.

Proof: To prove the theorem we show that for every $X_1^{n-1}, X_2^{n-1} \in (-\infty, \infty)$ and every $\alpha \in [0,1]$

$$B_n\left(\alpha X_1^{n-1} + (1-\alpha)X_2^{n-1}\right) \leq \alpha B_n\left(X_1^{n-1}\right) + (1-\alpha)B_n\left(X_2^{n-1}\right). \quad (32)$$

Let $X_1^{n-1}, X_2^{n-1} \in (-\infty, \infty)$ and $\alpha \in [0,1]$, it can straightforwardly be shown with the corresponding Hessian that G, as well as its expectation is convex in $(X^{n-1}, \int_{(n-1)\tau}^{t} u_n(s)ds)$, i.e.

$$\begin{cases} E\left[G\left(\alpha u_{n,1} + (1-\alpha)u_{n,2}, \alpha X_1^{n-1} + (1-\alpha)X_2^{n-1}\right)\right] \\ \quad \leq \alpha E\left[G\left(u_{n,1}, X_1^{n-1}\right)\right] + (1-\alpha)E\left[G\left(u_{n,2}, X_2^{n-1}\right)\right] \end{cases} \quad (33)$$

for every two single period policies $u_{n,1}, u_{n,2}$ and every two initial conditions X_1^{n-1}, X_2^{n-1}.

Now consider two production plans $\mathbf{u}^{n,1} = [u_{n,1}, u_{n+1,1}, \ldots, u_{N,1}]$ and $\mathbf{u}^{n,2} = [u_{n,2}, u_{n+1,2}, \ldots, u_{N,2}]$.

The combination $\alpha \mathbf{u}^{n,1} + (1-\alpha)\mathbf{u}^{n,1}$ is a production plan as well, and we have

$$\begin{cases} J_n\left(\alpha \mathbf{u}^{n,1} + (1-\alpha)\mathbf{u}^{n,2}, \quad \alpha X_1^{n-1} + (1-\alpha)X_2^{n-1}\right) \\ \quad = E_n\left[G\left(\alpha u_{n,1} + (1-\alpha)u_{n,2}, \right.\right. \\ \qquad \left.\left. \alpha X_1^{n-1} + (1-\alpha)X_2^{n-1}\right)\right] \\ \qquad + E\left[\sum_{k=n}^{N} G\left(\alpha u_{k+1,1} + (1-\alpha)u_{k+1,2}, \right.\right. \\ \qquad \left.\left. \alpha X_1^{k} + (1-\alpha)X_2^{k}\right)\right] \\ \quad \leq \alpha E_n\left[G\left(u_{n,1}, X_1^{n-1}\right)\right] + (1-\alpha)E_n\left[G\left(u_{n,2}, X_2^{n-1}\right)\right] \\ \qquad + E\left[\sum_{k=n}^{N}\left(\alpha G\left(u_{k+1,1}, X_1^k\right) + (1-\alpha)G\left(u_{k+1,2}, X_2^k\right)\right)\right] \\ \quad = \alpha J_n\left(\mathbf{u}^{n,1}, X_1^{n-1}\right) + (1-\alpha)J_n\left(\mathbf{u}^{n,2}, X_2^{n-1}\right). \end{cases}$$

Therefore we obtain

$$\begin{cases} \alpha B_n\left(X_1^{n-1}\right) + (1-\alpha)B_n\left(X_2^{n-1}\right) \\ \quad = \alpha \min_{\mathbf{u}^n} J_n\left(\mathbf{u}^n, X_1^{n-1}\right) \\ \qquad + (1-\alpha)\min_{\mathbf{u}^n} J_n\left(\mathbf{u}^n, X_2^{n-1}\right) \\ \quad = \alpha J_n\left(\mathbf{u}^{n^*}\left(X_1^{n-1}\right), X_1^{n-1}\right) \\ \qquad + (1-\alpha)J_n\left(\mathbf{u}^{n^*}\left(X_2^{n-1}\right), X_2^{n-1}\right) \\ \quad \geq J_n\left(\alpha \mathbf{u}^{n^*}\left(X_1^{n-1}\right) + (1-\alpha)\mathbf{u}^{n^*}\left(X_2^{n-1}\right), \right. \\ \qquad \left. \alpha X_1^{n-1} + (1-\alpha)X_2^{n-1}\right) \\ \quad \geq \min_{\mathbf{u}^n} J_n\left(\mathbf{u}^n, \alpha X_1^{n-1} + (1-\alpha)X_2^{n-1}\right) \\ \quad = B_n\left(\alpha X_1^{n-1} + (1-\alpha)X_2^{n-1}\right) \end{cases}$$

where $\mathbf{u}^{n^*}(x)$ denotes an optimal policy while the initial inventory is equal to x. ■

Similar to Lemmas 1–9, the optimal policies corresponding to the negative initial inventories are formulated, which along with Theorem 1 implies that the multi-period problem (1)–(4) has an optimal solution, $u^*(t)$, with at most two switching points and three production rates $\{0, b_n, U\}$. We next focus on the effect of the limited backlogs. Specifically, we show that in such a case it is no longer optimal to produce at the maximum rate U, as determined by the corresponding policies extracted from Lemmas 1–9 in Theorem 2.

Theorem 2: Assume that the sales are lost at the end of each period, $X^n = \max\{Y(n\tau) - D_n\tau, 0\}, 1 \leq n \leq N$. The optimal solution of the multi-period problem (1)–(4) is determined by (21), (22), (25) for the case of $t_4 = (N-1)\tau$ and (27).

VII. Conclusion

In this work we address a stochastic, optimal control problem of continuous inventory replenishment when the information on inventories is transmitted periodically. We derive the optimal policies which are classified into possible cases based on the level of the observed inventory. We show that the cost-to-go (Bellman) function is convex under any production policy. Furthermore, we prove that the optimal solution is piecewise constant with at most two switching points at each period. Specifically, the production (replenishment) rate at a period is always either equal to zero, or to the maximal possible rate, or to an intermediate value. The intermediate value is a parameter solely determined by the distribution of the demand at the period and the unit surplus/shortage costs. Moreover, if the sales are lost at the end of each period, the optimal production rate reduces to only two levels so that the production at a maximum rate is not optimal.

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