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Optimal scheduling of parallel machines with constrained resources

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Abstract

This paper analyzes a manufacturing system consisting of parallel machines, which produce one product-type with controllable production rates subject to continuously-divisible, time-dependent resources. The objective is to produce the required amount of product-type units by a due date while minimizing inventory, backlog and production related costs over a production horizon. With the aid of the maximum principle, a number of analytical rules of the optimal scheduling is derived whereby the continuous-time scheduling is reduced to discrete sequencing and timing. As a result, a polynomial-time algorithm is developed for solving the problem.

Keywords: Scheduling; Systems dynamics; Constrained resources

1. Introduction

The focus of this paper is a deterministic production system consisting of parallel machines which share resources to produce a number of units of the same product-type in response to demand. Similar to the systems considered by many authors in recent years [1–7], the system considered here includes a buffer with unlimited capacity placed after the machines for the product-type units. If cumulative production of the product exceeds cumulative demand, buffer carrying or inventory costs are incurred. If, on the other hand, cumulative demand exceeds cumulative production, backlog costs are incurred. In addition to inventory related costs, production costs are incurred if the machines are not idle. Given the constrained resources,

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the objective is to find production levels over time such that inventory, backlog as well as production costs are minimized.

One approach, modeling production systems as control problems, was introduced by Kimemia and Gershwin [3] for a system with random machine breakdowns. Two numerical approaches for similar deterministic systems have been developed in [5,7] to cope with continuous-time, multiple-machine production and project scheduling characterized by continuously-divisible constrained resources and arbitrary dynamic demands. Specifically, the former presents an algorithm for dynamic assignment while the latter discusses similarities with the resource constrained production and project scheduling models suggesting an adaptation and numerical method suitable for both production and project scheduling. These general approaches possess two major drawbacks. First, they are based on the gradient projection method to approximate the optimal solution in pseudo-polynomial time, thereby limiting the scale of the problems. Secondly, the accuracy of the solution found is very difficult to estimate and therefore to guarantee. As an alternative to the general approaches, increasing attention is being paid to special cases whereby production systems can be studied and solved analytically rather than numerically. These cases are usually characterized by a single machine producing several product-types given constant demand [1,4,2]. The properties of optimal schedules with continuously-divisible, doubly constrained resources were developed in [8] with the objective of minimizing project duration.

In this paper, unlike the above mentioned analytical works which relied on constant demand and levels of resource usage over time, demand and available resources are piece-wise constant. In other words, the product units are requested at one point in time, a due date, while the level of the available shared resource changes with time in an arbitrary, step-wise manner.

As an example of the real production system that can be modeled in this way, consider a typical fruit juice blending process. At the first stage of the process fruit concentrates are pumped into tanks and stirred. Next, the juice is pumped to continuous parallel blenders to add water, and in some cases sugar, to the concentrate. The output of the blenders is pumped into buffer tanks. There are several shared resources in the system. Many tanks share the same pipes, pumps, cleaning and processing equipment. Moreover, periodic cleaning operations, preventive maintenance and high priority orders make the availability of these resources time dependent. Therefore, production scheduling is of significant importance for such a system. It normally takes place on a weekly basis to meet juice demands as closely as possible by allocating blenders with respect to their utilization/production rates and available shared resource. This example will be further employed in the paper to illustrate the approach.

With the aid of the maximum principle, a number of analytical rules are derived for optimal selection of the machines, their production rates, sequencing and timing. Consequently, the continuous-time scheduling problem is reduced to a combinatorial search for a limited number of switching time points. As a result, the two drawbacks mentioned above are overcome: the solution is obtained in strictly polynomial time and its accuracy is guaranteed. Special cases, when the switching points can be located analytically rather than combinatorially are also discussed and the complexity estimates are derived.

2. Statement of the problem

Consider a production system which produces units of a single product-type in response to demand $D(t)$ for this product-type. The production system consists of N parallel machines and a buffer placed after the machines to hold completed units of the product-type. The maximal production rate of machine n is denoted as U_n. This system represents a cumulative flow of the product-type units through the machines and the buffer:

$$\dot{X}(t) = \sum_n U_n u_n(t), \quad X(0) = 0, \tag{1}$$

where $X(t)$ is the amount of the product-type units produced by time t (state variable) and $u_n(t)$ is the relative production rate (loading) of machine n at time t or production decision variable,

$$0 \leqslant u_n(t) \leqslant 1, \quad n = 1, 2, \ldots, N. \tag{2}$$

Although all machines can operate concurrently, the number of the machines which produce the product-type is subject to resource availability at every time point:

$$\sum_n u_n(t) \leqslant R(t), \tag{3}$$

where $R(t)$ is the maximal total processing rate determined by the availability of various categories of resources at time t. To simplify analysis of the problem, we assume that $R(t)$ is a piece-wise constant function over K constant intervals:

$$\begin{aligned} &R(t) = r_k, \quad t_{k-1} \leqslant t < t_k, \quad k = 1, \ldots, K-1, \\ &R(t) = r_K, \quad t_{K-1} \leqslant t \leqslant t_K = T, \quad t_0 = 0. \end{aligned} \tag{4}$$

For example, consider a fruit juice blending system that consists of five blenders followed by interconnected buffer tanks. The maximal blending rates of these parallel production lines are $U_1 = 80$, $U_2 = 100$, $U_3 = 70$, $U_4 = 60$, $U_5 = 90$ l per hour. The planning horizon is 200 hours. Predetermined periodic cleaning of pipes will disrupt the production during this particular planning horizon so that the total amount of the juice which can be pumped through the system is reduced to 300 l per hour during the first 80 hours and to 350 l per hour during the remaining 120 hours. If priorities to the blenders are 1-3-4-5-2, the disruption implies $r_1 = 4$ and $r_2 = 4 + \frac{80+100+70+60+90-350}{100} = 4.5$. Thus, expressions (1) and (4) take the following form:

$$\dot{X}(t) = 80u_1(t) + 100u_2(t) + 70u_3(t) + 60u_4(t) + 90u_5(t),$$

$$R(t) = \begin{cases} 4 & \text{for } 0 \leqslant t < 80; \\ 4.5 & \text{for } 80 \leqslant t \leqslant 120. \end{cases}$$

Demand for the product-type $D(t)$ is cumulative and is characterized by amount D of the product-type units to be produced by due date t_d:

$$D(t) = \begin{cases} D & \text{if } t \geqslant t_d; \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Then $X(t) - D(t)$ is the product surplus level at the buffer at time t when $X(t) \geqslant D(t)$ and the backlog level otherwise.

The objective is to find rates $u_n^*(t)$ that satisfy constraints (1)–(3) while minimizing the following linear cost functional over planning horizon T:

$$\int_0^T \left(c_x^+ X^+(t) + c_x^- X^-(t) + \sum_n c_u^n u_n(t) \right) \mathrm{d}t \to \min, \tag{6}$$

where c_x^+ represents the product-type unit holding cost, c_x^- is the product-type unit backlog cost, c_u^n is the production cost of machine n (all costs are per time unit) and

$$X^+(t) = \max\{0, X(t) - D(t)\} \quad \text{and} \quad X^-(t) = \max\{0, D(t) - X(t)\}.$$

3. The dual problem

To study problem (1)–(6), we formulate a dual problem with co-state variable $\psi(t)$ satisfying the following co-state (dual) equation with transversality (boundary) condition:

$$\dot{\psi}(t) = \begin{cases} c_x^+ & \text{if } X(t) > D(t), \\ -c_x^- & \text{if } X(t) < D(t), \quad \psi(T) = 0, \\ c \in [-c_x^-, c_x^+] & \text{if } X(t) = D(t). \end{cases} \tag{7}$$

The co-state (dual) variable, $\psi(t)$, measures the dynamic marginal cost which is the change in the objective function value resulting from a unit change of the inventory level at time t, $X(t)$.

The Hamiltonian is the objective for the dual problem, which is maximized at every point of time by the optimal production rates according to the maximum principle [6]:

$$H(t) = -c_x^+X^+(t) - c_x^-X^-d(t) - \sum_n c_u^n u_n(t) + \psi(t)\sum_n U_n u_n(t) \to \max \tag{8}$$

subject to constraints (2) and (3).

4. Properties of the optimal solution

According to the maximum principle, in order to identify properties of the optimal solution, we consider only production rate-dependent terms of the Hamiltonian:

$$H^u(t) = \psi(t)\sum_n U_n u_n(t) - \sum_n c_u^n u_n(t) = \sum_n u_n(U_n\psi(t) - c_u^n) \to \max. \tag{9}$$

It is easy to observe from (9) that if for a particular machine n

$$U_n\psi(t) - c_u^n < 0, \tag{10}$$

then $u_n(t) = 0$ maximizes the Hamiltonian (*No-Production Rule*). Moreover the same rule is true when there exists a machine for which

$$U_n\psi(t) - c_u^n = 0 \tag{11}$$

holds at an interval of time as shown in the following lemma.

Lemma 1. *Given problem (1)–(6), if there exists a machine n for which $U_n\psi(t) - c_u^n = 0$ holds at an interval of time, then $u_n(t) = 0$ over this interval.*

Proof. By differentiating equality (11) over the given interval of time we find $\dot{\psi}(t) = 0$, which due to conditions (7) can hold only if $X(t) = D(t)$ at the interval. Consequently, by taking into account the demand form (5) and production equation (1), we find that $X(t) = D(t)$ can be supported at the interval only if $u_n(t) = 0$. Thus, if there exists a machine n for which $U_n\psi(t) - c_u^n = 0$ holds at an interval of time, then $u_n(t) = 0$ over this interval. □

Conditions (10) and (11) present the no-production rule when $U_n\psi(t) - c_u^n \leqslant 0$. However, when this is not the case, i.e. $U_n\psi(t) - c_u^n > 0$, an important production rule can be established. To formalize this rule, we denote the gradient of the Hamiltonian with respect to the production rate

$$G_n(t) = \frac{\partial H(t)}{\partial u_n(t)} = U_n\psi(t) - c_u^n \tag{12}$$

and substitute it into (9). Consequently, the dual objective takes the following form:

$$H^u(t) = \sum_n G_n(t)u_n(t) \to \max. \tag{13}$$

The optimal solution for problem (13), (2) and (3) is known to exhibit the so-called greedy property. To be specific, consider a couple of positive coordinates of the Hamiltonian gradient which are not equal to one another $G_n(t) > G_{n'}(t)$ and are greater than the other coordinates. Then the production rate-dependent term of the Hamiltonian is maximized if machine n, which is characterized by the maximal gradient, produces so as to best utilize the resource:

$$u_n(t) = 1 \quad \text{if } R(t) \geqslant 1 \ (u_n(t) = R(t) \quad \text{if } R(t) < 1 \text{ and } u_{n'}(t) = 0).$$

Consequently, the other machine, n', greedily contends for the remaining resource, i.e.

$$u_{n'}(t) = 1 \quad \text{if } 1 \leqslant R(t) < 2 \text{ and } u_{n'}(t) = R(t) - 1.$$

Similarly, one can continue with setting optimal rates if there are sufficient resources, $R(t) > 2$, and more than a couple of machines with positive gradients such that they will utilize the remaining capacity with priorities determined by the values of their gradients.

The *Greedy Production Rule* is summarized as follows. Given $G_n(t) > G_{n'}(t)$, $G_n(t) > 0$, $\forall n' \neq n$, the following conditions hold:

$$\begin{aligned} &u_n(t) = 1 && \text{if } R(t) \geqslant 1, \quad u_n(t) = R(t) \quad \text{if } R(t) < 1, \\ &u_{n'}(t) = 1 && \text{if } |A(n'')| + 1 \leqslant R(t) \text{ and } G_{n'}(t) > 0, \\ &u_{n'}(t) = R(t) - |A(n'')| && \text{if } 1 + |A(n'')| > R(t), \quad G_{n'}(t) > 0 \text{ and } |A(n'')| < R(t), \\ & && \text{otherwise } u_{n'}(t) = 0, \end{aligned} \tag{14}$$

where $A(n'') = \{n'' | G_{n''} > G_{n'}\}$.

Note, that according to the no-production and greedy production rules, if point t' is the point where machine n switches on (off) after (before) machine n', the following holds:

$$\begin{aligned} &\text{either } G_n(t) < 0 \quad \text{and} \quad G_{n'}(t) > 0 \quad \text{for } t < t', \text{ that is } \frac{c_u^{n'}}{U_{n'}} < \psi(t) < \frac{c_u^n}{U_n}; \\ &\text{or } G_{n'}(t) > G_n(t) > 0 \quad \text{for } t < t' \quad \text{and} \quad G_n(t) > G_{n'}(t) > 0 \quad \text{for } t > t', \end{aligned} \tag{15}$$

that by taking into account (12) means $\frac{c_u^n}{U_n} > \frac{c_u^{n'}}{U_{n'}}$. This in turn implies the following *priority rule*: if for a couple of machines n and n', machine n switches on (off) after (prior to) machine n' then $\frac{c_u^n}{U_n} > \frac{c_u^{n'}}{U_{n'}}$.

Henceforth, without loss of generality we assume that all machines are sorted and numbered with respect to the priority rule, that is

$$\frac{c_u^1}{U_1} \leqslant \frac{c_u^2}{U_2} \leqslant \cdots \leqslant \frac{c_u^N}{U_N}. \tag{16}$$

Given the no-production and greedy production rules, the following lemma proves that either the production is not profitable at all, or once the production of the product-type units starts it can be completely halted only if the resource is not available.

Lemma 2 (Non-Preemptive Production Rule). *Given problem* (1)–(6),

- *if* $T - t_d \leqslant \frac{c_u^1}{c_x^- U_1}$, *the optimal solution is no production at all,*
- *otherwise, if there exists a point of time* $t_f < T - \frac{c_u^1}{c_x^- U_1}$, *such that* $R(t_f) > 0$ *and* $R(t) = 0$ *for* $t_f < t \leqslant T - \frac{c_u^1}{c_x^- U_1}$, *then production necessarily starts and proceeds according to the greedy production rule until either the demand is completed or until point* $t_f < T - \frac{c_u^1}{c_x^- U_1}$.

Proof. First, consider the case that no production is the rule applicable along the entire planning horizon, i.e.

$$u_n(t) = 0 \quad \text{and} \quad X(t) = 0 \quad \text{for } n = 1, 2, \ldots, N \quad \text{and} \quad 0 \leqslant t \leqslant T. \tag{17}$$

Then the following behavior of the co-state variable evidently satisfies the dual equation (7):

$$\psi(t) = c_x^-(T - t_d) \quad \text{for } 0 \leqslant t \leqslant t_d, \quad \psi(t) = c_x^-(T - t) \quad \text{for } t_d \leqslant t \leqslant T \tag{18}$$

and the maximum principle, if $U_n\psi(t) - c_u^n \leqslant 0$ for $n = 1, 2, \ldots, N$ and $0 \leqslant t \leqslant T$ (see Lemma 1). With respect to (18), the last inequality implies

$$\psi(t) = c_x^-(T - t) \leqslant \min_n \frac{c_u^n}{U_n} = \frac{c_u^1}{U_1} \quad \text{for } t_d \leqslant t \leqslant T. \tag{19}$$

Thus, if inequality (19) is true for the marginal case of $t = t_d$, no production is the only feasible solution which satisfies the maximum principle over the entire planning horizon, as stated in the first condition of this lemma. However, if condition (19) does not hold for $t = t_d$, then there necessarily exists an interval of time and a machine, n, such that $G_n(t) > 0$, i.e., the greedy production rule is optimal at this interval as stated in the second condition of the lemma. Note, even though the production is optimal at an interval of time, according to the greedy production rule it can be carried out only if the resource is available at least at some point in the interval, $R(t) > 0$. Therefore, we distinguish between the no-production rule which may become optimal even if the resource is available and the greedy production rule at zero rate which may occur if no resource is available.

Finally, we prove by contradiction that once production starts, the greedy production rule remains optimal until either the demand is satisfied or until $\bar{t}_2 = T - \frac{c_u^1}{c_x^- U_1}$. Indeed, if the production starts at a point, t_s, inventories can only increase prior to the due date, which with respect to demand form (5) implies $X(t) > D(t)$ for $t_s < t < t_d$ and $X(t) < D(t)$ either for $t_d \leqslant t < t_f$, if the demand is satisfied at a point $t_f < T - \frac{c_u^1}{c_x^- U_1}$, or for $t_d \leqslant t \leqslant T$. This inventory behavior along with co-state equation (7) implies:

$$\dot{\psi}(t) > 0 \quad \text{for } t_s < t < t_d \quad \text{and} \quad \dot{\psi}(t) < 0 \quad \text{for } t_d < t \leqslant T. \tag{20}$$

Let there exist an interval of time $[t_x, t_y]$ where the demand is not fulfilled and the no-production rule interrupts greedy production, i.e. with respect to (19):

$$\psi(t) < \min_n \frac{c_u^n}{U_n} = \frac{c_u^1}{U_1} \quad \text{for } t_x \leqslant t \leqslant t_y, \tag{21}$$

$$\psi(t) > \min_n \frac{c_u^n}{U_n} = \frac{c_u^1}{U_1} \quad \text{for } t_y < t \quad \text{and} \quad t_s \leqslant t < t_x. \tag{22}$$

Then the inequalities (21) and (22) for $t_s \leqslant t < t_x$ can be met if $\psi(t)$ is a decreasing function at interval $t_s \leqslant t < t_x$ which according to (20) is only possible after the due date, i.e. $t_s \geqslant t_d$. On the other hand, in order for (22) to be met when $t > t_y$, $\psi(t)$ must increase from a point t_z, $t_x < t_z < t_y$, that is $\dot{\psi}(t) > 0$ for $t > t_d$ which contradicts (20). □

Corollary 1 (Sequencing Rule). *Given problem (1)–(6) and* $T - t_d > \frac{c_u^1}{c_x^- U_1}$

- *there can be at most two switching time points* $\bar{t}_1$ *and* $\bar{t}_2$, $0 \leqslant \bar{t}_1 \leqslant t_d$, $t_d \leqslant \bar{t}_2 \leqslant T - \frac{c_u^1}{c_x^- U_1}$ *where production rules are changed over in accordance with the following sequence: no-production, greedy production, no-production;*
- *the co-state variable is bounded as* $\frac{c_u^1}{U_1} \leqslant \psi(t) \leqslant c_x^-(T - t_d)$ *for* $0 \leqslant t \leqslant \bar{t}_1$.

Proof. The optimal sequencing of the production rules with $0 \leqslant \bar{t}_1 \leqslant t_d$, $t_d < \bar{t}_2 \leqslant T - \frac{c_u^1}{c_x^- U_1}$, immediately follows from Lemma 2. Indeed if preemption is not optimal during greedy production, then the only sequence of no-production and greedy production rules which can be applied is that stated in the corollary. Evidently, point $\bar{t}_1$ may equal zero implying that the initial no production vanishes from the sequence. Then the maximal value the co-state variable can achieve is determined by the transversality condition $\psi(T) = 0$ and the maximal rate this condition can attain $\dot{\psi}(t) = -c_x^-$. Thus, given the sequence, the upper and lower bounds for the co-state variable are readily obtained from (18) and (22) respectively. □

According to Lemma 2, production is profitable if $T - t_d > \frac{c_u^1}{c_x^- U_1}$. According to Corollary 1, production should begin from time point $\bar{t}_1 0$ and continue until point $\bar{t}_2$, $\bar{t}_1 < \bar{t}_2 \leqslant T - \frac{c_u^1}{c_x^- U_1} < T$. Furthermore, optimal production rate at each point $t, \bar{t}_1 \leqslant t < \bar{t}_2$, is determined by the greedy production rule.

Given the optimal sequencing rule, we can now elaborate on the subject of optimal timing. This is accomplished by a constructive approach in Lemmas 3–5. Specifically, we separate the analysis for two different cases of pressing and loose planning horizons. Based on the derived rules, a feasible solution is constructed for each case so that all state and co-state constraints are met. Then it is verified whether the constructed solution maximizes the Hamiltonian.

Note, statements (1)–(6) do not impose on the optimal production to necessarily satisfy the demand. This fact, turns out, highly influences the complexity of timing. The following lemma discusses the case that occurs when the time point fixing the end of the greedy production is marginal, $\bar{t}_2 = T - \frac{c_u^1}{c_x^- U_1}$. This implies that if the production is organized optimally, the planning horizon is so pressing that not only is the production late with respect to the due date, but the entire planning horizon is insufficient to fully satiate the demand as illustrated in Fig. 1. This case is characterized by two different solutions which can be uniquely identified if $\bar{t}_1$ and $\psi(\bar{t}_1)$ (and therefore $G_n(\bar{t}_1), n = 1, \ldots, N$) are determined.

Lemma 3 (Pressing Horizon Case). *Given problem (1)–(6), $T - t_d > \frac{c_u^1}{c_x^- U_1}$ and $X(T) < D$, the optimal solution is $u_n^*(t) = 0$, for n = 1,2,. . .,N, $0 \leqslant t < \bar{t}_1$ and $\bar{t}_2 \leqslant t \leqslant T$, $U_n^*(t)$, satisfy* (14) *for n = 1,2,. . .,N and $\bar{t}_1 \leqslant t < \bar{t}_2$, where $\bar{t}_2 = T - \frac{c_u^1}{c_x^- U_1}$, and*

- *if $t_d - \frac{c_x^-}{c_x^+}(T - t_d) \geqslant -\frac{c_u^1}{c_x^+ U_1}$ then $\bar{t}_1 = t_d - \frac{c_x^-}{c_x^+}(T - t_d) + \frac{c_u^1}{c_x^+ U_1}$ and $\psi(\bar{t}_1) = \frac{c_u^1}{U_1}$;*
- *otherwise $\bar{t}_1 = 0$ and $\psi(0) = c_x^-(T - t_d) - c_x^+ t_d$.*

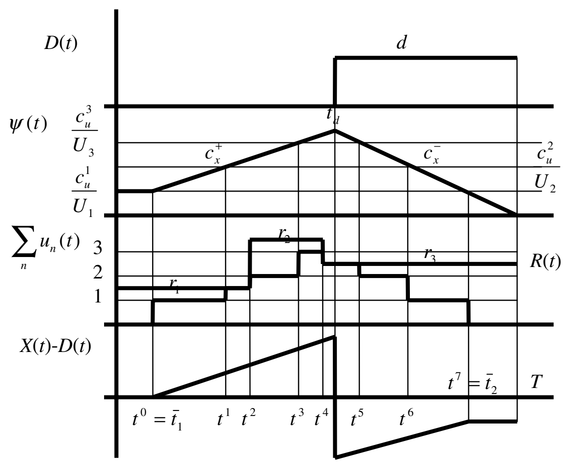

Fig. 1. Typical system dynamics for the case of the pressing planning horizon.

Proof. To prove this lemma, we construct a solution for the state and co-state variables which satisfies the maximum principle (see Fig. 1), that is the derived production rules, Lemmas 1–3 and co-state equation (7). Consequently, if such a solution is feasible, then it is also optimal.

Let the production rules be applied in accordance with the sequencing rule: no-production, greedy production, no-production with points $\bar{t}_1$ and $\bar{t}_2$ representing the switches. Then the following solution for the co-state variable,

$$\begin{aligned} &\psi(t) = \frac{c_u^1}{U_1} \quad \text{for } 0 \leqslant t \leqslant \bar{t}_1; \\ &\psi(t) = c_x^-(T - t_d) + c_x^+(t - t_d) \quad \text{for } \bar{t}_1 \leqslant t \leqslant t_d; \\ &\psi(t) = c_x^-(T - t) \quad \text{for } t_d \leqslant t \leqslant T \end{aligned} \tag{23}$$

evidently satisfies dual equation (7) and Lemmas 1 and 2, if $X(T) < D$ and $\bar{t}_2 = T - \frac{c_u^1}{c_x^- U_1}$. Thus, solution $u_n^*(t) = 0$ for $n = 1, 2, \ldots, N$, $0 \leqslant t < \bar{t}_1$ and $\bar{t}_2 \leqslant t \leqslant T$, $u_n^*(t)$ satisfying (14) for $n = 1, 2, \ldots, N$ and $\bar{t}_1 \leqslant t < \bar{t}_2$ is optimal when Eqs. (23) are feasible, i.e. $\bar{t}_1$ determined from (23) is non-negative. From the first two equations of (23) at point $\bar{t}_1$, we find:

$$\psi(\bar{t}_1) = \frac{c_u^1}{U_1} = c_x^-(T - t_d) + c_x^+(\bar{t}_1 - t_d). \tag{24}$$

By solving Eq. (24) with respect to $\bar{t}_1$ and requiring $\bar{t}_1 \geqslant 0$ we obtain the first condition stated in this lemma.

To prove the second condition, we consider the special case of the general sequence when the no-production rule is not applicable prior to greedy production, i.e. $\bar{t}_1 = 0$, $u_n^*(t)$ satisfy (14) for $n = 1, 2, \ldots, N$ and $0 \leqslant t < \bar{t}_2$, $u_n^*(t) = 0$ for $n = 1, 2, \ldots, N$ and $\bar{t}_2 \leqslant t \leqslant T$. Then we find the following solution for the co-state variable,

$$\begin{aligned} &\psi(t) = \psi(0) + c_x^+ t \quad \text{for } 0 \leqslant t \leqslant t_d; \\ &\psi(t) = c_x^-(T - t) \quad \text{for } t_d \leqslant t \leqslant T \end{aligned} \tag{25}$$

satisfies co-state equation (7) and Lemmas 1, 2, if $X(T) < D$ and $\psi(0) > \frac{c_u^1}{U_1}$. Consequently, the second condition of the lemma is verified by solving (25) at point $t = t_d$ with respect to $\psi(0)$. □

According to Lemma 3, given $T - t_d > \frac{c_u^1}{c_x^- U_1}$ and $X(T) < D$, the optimal solution is $u_n^*(t) = 0$ for $0 \leqslant t < \bar{t}_1$ and $\bar{t}_2 \leqslant t \leqslant T$.With respect to (23) and the greedy production rule, this implies that if there are enough resources, then the ith machine, $i > L$, can switch on at time point

$$\frown{t}_i = \bar{t}_1 + \frac{c_u^i}{c_x^+ U_i} - \frac{\bar{\psi}}{c_x^+}, \quad \frown{t}_i \leqslant t_d, \tag{26}$$

where

$$\bar{\psi} = \psi(0), \quad \frac{c_u^L}{U_L} \leqslant \psi(0) < \frac{c_u^{L+1}}{U_{L+1}}$$

if $\bar{t}_1 = 0$ and $\bar{\psi} = \frac{c_u^1}{U_1}$, $L = 1$, if $\bar{t}_1 > 0$, and the ith machine, $i \geqslant 1$, can switch off at point

$$\smile{t}_i = \bar{t}_2 + \frac{c_u^i}{c_x^- U_i} - \frac{c_u^1}{c_x^- U_1}, \quad \smile{t}_i > t_d. \tag{27}$$

Note that time points (26) and (27) depend on $\bar{t}_2$ and either $\bar{t}_1$ or $\overline{\psi}$. Furthermore, with respect to (4), the maximal availability of resources can impose the additional switching points,

$$t_k, \quad k = 1, \ldots, K. \tag{28}$$

We, thus, conclude with the following switching rule.

Lemma 4 (Switching Rule). *Given problem (1)–(6),* $T - t_d > \frac{c_u^1}{c_x^- U_1}$, *time points* $\bar{t}_1, \bar{t}_2$, $0 \leqslant \bar{t}_1 \leqslant t_d$ *and* $t_d < \bar{t}_2 < T$, *where the production rules change over, the optimal production rate is a piece-wise constant function with at most* $2N + K$ *switching points.*

Proof. The proof immediately follows from the greedy production rule and the facts that $R(t)$ is a piece-wise constant function of time and the co-state variable is linear in time. □

Lemma 4 shows that there is no need to apply the greedy production rule at every point of time. This implies that the interval $\bar{t}_1 \leqslant t \leqslant \bar{t}_2$ can be divided into a limited number of subintervals within each of which the production rate does not re-switch, i.e., the greedy production rule is applied once for each sub-interval. To make use of this switching rule, let us sort time points (26)–(28) in ascending order, renumber them in this order and combine into an ordered set, $s(\bar{t}_1, \bar{t}_2, \overline{\psi})$. Then the size of this is $|s(\bar{t}_1, \bar{t}_1, \overline{\psi})| \leqslant 2N + K$. Furthermore, for each time point t_j, $j = 1, 2, \ldots, |s(\bar{t}_1, \bar{t}_2, \overline{\psi})|$, we use the following information:

- $Z(j)$ number of machines which can work with respect to (26) and (27);
- $Q(j)$ time point origin, $Q(j) = 0,1$ or 2 if point t_j is defined by (26), (27) or (28), respectively;
- $Q^0(j)$ time point original index if $Q(j) = 0$;
- $Q^1(j)$ time point original index if $Q(j) = 1$.

Then

$$t_j = \bar{t}_1 + \frac{c_u^{Q^0(j)}}{c_x^+ U_{Q^0(j)}} - \frac{\overline{\psi}}{c_x^+} \quad \text{if } Q(j) = 0$$

and

$$t_j = \bar{t}_2 - \frac{c_u^{Q^1(j)}}{c_x^- U_{Q^1(j)}} + \frac{c_n^1}{c_x^- U_1} \quad \text{if } Q(j) = 1$$

and the greedy production rule takes the following explicit form:

$$u_n(t_j) = \begin{cases} 1 & \text{if } n \leqslant \min\{R(t_j), Z(j)\}; \\ n - R(t_j) & \text{if } R(t_j) < n < \min\{R(t_j) + 1, Z(j)\}; \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

and

$$u_n^*(t) = u_n(t_j) \quad \text{for } t_j \leqslant t < t_{j+1}, \quad j = 1, 2, \ldots, |s(\bar{t}_1, \bar{t}_2, \overline{\psi})|. \tag{30}$$

Thus, given (29), the terminal inventory is determined as

$$X(T) = X(\bar{t}_2) = \sum_{j=1}^{|s(\bar{t}_1, \bar{t}_2, \overline{\psi})|} \sum_{n=1}^{N} U_n u_n(t_j)(t_{j+1} - t_j). \tag{31}$$

The following lemma uses (31) to treat the case when the production horizon is not pressing.

Lemma 5 (Loose Horizon Case). *Given problem (1)–(6), $T - t_d > \frac{c_u^1}{c_x^- U_1}$, $X(T) = D$, the optimal solution is $u_n^*(t) = 0$ for $0 \leqslant t < \bar{t}_1$ and $\bar{t}_2 \leqslant t \leqslant T$, $u_n^*(t)$ satisfy (14) for $\bar{t}_1 \leqslant t < \bar{t}_2$,*

- *if*

$$\sum_{j=1}^{\left|s\left(0,\left(1+\frac{c_x^+}{c_x^-}\right)\frac{c_u^1}{U_1}\right)\right|} \sum_{n=1}^{N} U_n u_n(t_j)(t_{j+1} - t_j) > D$$

then $\bar{t}_1$ and $\bar{t}_2$ satisfy the following system of equations:

$$\sum_{j=1}^{|s(\bar{t}_1,\bar{t}_2,\bar{\psi})|} \sum_{n=1}^{N} U_n u_n(t_j)(t_{j+1} - t_j) - D = 0, \quad \bar{t}_2 = -\frac{c_x^+}{c_x^-}(\bar{t}_1 - t_d) + t_d,$$

where

$$\bar{\psi} = \frac{c_u^1}{U_1}, \quad t_j = \bar{t}_1 + \frac{c_u^{Q^0(j)}}{c_x^+ U_{Q^0(j)}} - \frac{c_n^1}{c_x^+ U_1} \quad \textit{if } Q(j) = 0$$

and

$$t_j = \bar{t}_2 - \frac{c_u^{Q^1(j)}}{c_x^- U_{Q^1(j)}} + \frac{c_n^1}{c_x^- U_1} \quad \textit{if } Q(j) = 1;$$

- *otherwise $\bar{t}_1 = 0$, $\bar{t}_2$ and $\psi(0)$ satisfy the following system of equations:*

$$\sum_{j=1}^{|s(0,\bar{t}_2,\psi(0))|} \sum_{n=1}^{N} U_n u_n(t_j)(t_{j+1} - t_j) - D = 0$$

and

$$\bar{t}_2 = \frac{\psi(0)}{c_x^-} + \left(1 + \frac{c_x^+}{c_x^-}\right) t_d - \frac{c_u^1}{c_x^- U_1},$$

where

$$t_1 = 0, \quad t_j = \frac{c_u^{Q^0(j)}}{c_x^+ U_{Q^0(j)}} - \frac{\psi(0)}{c_x^+} \quad \textit{for } j > 1 \quad \textit{if } Q(j) = 0$$

and

$$t_j = \bar{t}_2 - \frac{c_u^{Q^1(j)}}{c_x^- U_{Q^1(j)}} + \frac{c_n^1}{c_x^- U_1} \quad \textit{if } Q(j) = 1.$$

Proof. To prove this lemma, we again construct a solution which satisfies the maximum principle (see Fig. 2) and study its feasibility. From Lemma 3 it follows that if the planning horizon is not pressing, then the following boundary conditions hold:

$$\psi(\bar{t}_1) + c_x^+(\bar{t}_1 - t_d) = c_x^-(\bar{t}_2 - t_d) + \frac{c_u^1}{U_1}, \tag{32}$$

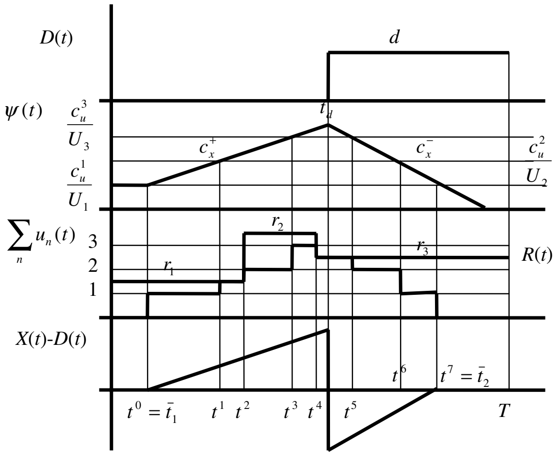

Fig. 2. Typical system dynamics for the case of the loose planning horizon.

$$X(\overline{t}_2) = D, \tag{33}$$

where either $\overline{t}_1 > 0$ and therefore $\psi(t) = \frac{c_u^1}{U_1}$ for $0 \leqslant t \leqslant \overline{t}_1$, or $\overline{t}_1 = 0$ and therefore $\psi(0) > \frac{c_u^1}{U_1}$. Thus, in either of these two cases we have exactly two unknowns $\overline{t}_1, \overline{t}_2$ or $\psi(0), \overline{t}_2$ which are defined by two equations (32) and (33). In other words, using (31), (33) takes the form stated in the lemma.

Finally, to distinguish between the two cases:

(i) $\overline{t}_1 > 0$, $\psi(t) = \frac{c_u^1}{U_1}$ for $0 \leqslant t \leqslant \overline{t}_1$;

(ii) $\overline{t}_1 = 0$, $\psi(0) > \frac{c_u^1}{U_1}$,

we consider the marginal case $\overline{t}_1 = 0$ and $\psi(0) = \frac{c_u^1}{U_1}$. By verifying $X(\overline{t}_2) \geqslant D$ when $\overline{t}_1 = 0$ and $\psi(0) = \frac{c_u^1}{U_1}$ we ensure the first condition of Lemma 5. If however $X(\overline{t}_2) < D$ when $\overline{t}_1 = 0$ and $\psi(0) = \frac{c_u^1}{U_1}$, the second statement immediately becomes feasible as stated in this lemma. □

Note that Lemma 3 identifies triples $\overline{t}_1, \overline{t}_2, \overline{\psi}$ independently of the other switching points defined by (26) and (27). Therefore, the problem is immediately solved when the production horizon is pressing. On the other hand, finding $\overline{t}_1$ from the linear equation,

$$\sum_{j=1}^{|s(\overline{t}_1,\overline{t}_2,\overline{\psi})|} \sum_{n=1}^{N} U_n u_n(t_j)(t_{j+1} - t_j) - D = 0 \tag{34}$$

derived in Lemma 5 is not immediate because the order of the time points t_j of the set $s(\overline{t}_1, \overline{t}_2, \overline{\psi})$ depends on $\overline{t}_1$. Therefore, we first need to identify time intervals τ_i, $i = 1, 2, \ldots, |s(\overline{t}_1, \overline{t}_2, \overline{\psi})|$, so that the order of the time points t_j of $s(\overline{t}_1, \overline{t}_2, \overline{\psi})$ does not change. This is accomplished by setting $\overline{\psi} = \frac{c_u^1}{U_1}$ and $\overline{t}_1$ such that $\overline{t}_2 = -\frac{c_x^+}{c_x^-}(\overline{t}_1 - t_d) + t_d = T - \frac{c_u^1}{c_x^- U_1}$ (see the first statement of Lemma 5) and constructing the corresponding $s(\overline{t}_1, \overline{t}_2, \overline{\psi})$. Next we find the shortest time interval $\tau_1 = \min_j\{t_{j+1} - t_j\}$ and assign $\overline{t}_1 := \overline{t}_1 + \tau_1$. Continuing this way until the entire production horizon is covered, we identify at most $2N + K$ time intervals and the corresponding sets $s(\overline{t}_1, \overline{t}_2 \overline{\psi})$. Since we do not know in advance, which interval unknown $\overline{t}_1$ belongs to, we will have to solve Eq. (34) repeatedly up to $2N + K$ times assuming that $\overline{t}_1$ belongs to each of the found intervals separately.

The same procedure can be employed to identify the co-state intervals ψ_i, $i = 1, 2, \ldots, |s(\bar{t}_1, \bar{t}_2, \bar{\psi})|$, so that the order of the time points t_j of the set $s(\bar{t}_1, \bar{t}_2, \bar{\psi})$ does not change. These intervals are needed to find $\psi(0)$ from the linear equation

$$\sum_{j=1}^{|s(0,\bar{t}_2,\psi(0))|} \sum_{n=1}^{N} U_n u_n(t_j)(t_{j+1} - t_j) - D = 0. \tag{35}$$

Specifically, by setting $\bar{t}_1 = 0$, $\bar{\psi} = \psi(0)$, $\psi(0) = \frac{c_u^1}{U_1}$ and $\bar{t}_2 = \frac{\psi(0)}{c_x^-} + \left(1 + \frac{c_x^+}{c_x^-}\right) t_d - \frac{c_u^1}{c_x^- U_1}$ (see the second statement of Lemma 5), we construct the corresponding $s(\bar{t}_1, \bar{t}_2, \bar{\psi})$. Next we find the shortest time interval $\tau_1 = \min_j\{t_{j+1} - t_j\}$, set $\psi = \psi(0) + c_x^+(0 + \tau_1)$ and $\bar{\psi} = \psi$. Then the corresponding co-state interval is $\psi_1 = \left[\frac{c_u^1}{U_1}, \psi\right]$. Continuing this way until the upper bound of the co-state variable is reached (see Corollary 1), we identify at most $2N + K$ co-state intervals and the corresponding sets $s(\bar{t}_1, \bar{t}_2, \bar{\psi})$. The described combinatorial approach is employed to cope with loose production conditions and is summarized in the next section.

5. Scheduling algorithm and complexity

In this section, a fast algorithm for continuous time scheduling is presented. The algorithm is based on Lemmas 1–5 and Corollary 1. It will find the exact optimal solution for both pressing and loose production conditions. The algorithm is as follows:

Step 1. Sort machines $n = 1, 2, \ldots, N$ by $\frac{c_u^n}{U_n}$ in non-decreasing order, renumber them and store the obtained sequence.

Step 2. If $T - t_d \leqslant \frac{c_u^1}{c_x^- U_1}$, set the optimal production rate as $u_n^*(t) = 0$ for $n = 1, \ldots, N$, $0 \leqslant t \leqslant T$; STOP. Otherwise go to the next step.

Pressing Planning Horizon Case

Step 3. Calculate $\bar{t}_2 = T - \frac{c_u^1}{c_x^- U_1}$. If $t_d - \frac{c_x^-}{c_x^+}(T - t_d) \geqslant -\frac{c_u^1}{c_x^+ U_1}$ then calculate $\bar{t}_1 = t_d - \frac{c_x^-}{c_x^+}(T - t_d) + \frac{c_u^1}{c_x^+ U_1}$ and $\bar{\psi} = \frac{c_u^1}{U_1}$. Otherwise go to Step 6.

Step 4. Calculate switching points by (26) and (27). Construct $s(\bar{t}_1, \bar{t}_2, \bar{\psi})$, $Q(j)$, $Q^0(j)$, $Q^1(j)$ and $Z(j)$, $j = 1, 2, \ldots, |s(\bar{t}_1, \bar{t}_2, \bar{\psi})|$.

Step 5. Calculate $u_n(t_j)$ by (29) for $j = 1, 2, \ldots, |s(\bar{t}_1, \bar{t}_2, \bar{\psi})|$ and $n = 1, \ldots, N$. If $\sum_{j=1}^{|s(\bar{t}_1,\bar{t}_2,\bar{\psi})|} \sum_{n=1}^{N} U_n u_n(t_j)(t_{j+1} - t_j) < D$, then set the optimal solution as: $u_n^*(t) = 0$ for $0 \leqslant t < \bar{t}_1$ and $\bar{t}_2 \leqslant t \leqslant T$, $u_n^*(t) = u_n(t_j)$ for $t_j \leqslant t < t_{j+1}$, $j = 1, 2, \ldots, |s(\bar{t}_1, \bar{t}_2, \bar{\psi})| - 1$, $n = 1, \ldots, N$. STOP. Otherwise go to Step 7.

Step 6. Set $\bar{t}_1 = 0$, $\psi(0) = c_x^-(T - t_d) - c_x^+ t_d$, $\bar{\psi} = \psi(0)$ and go to Step 4.

Loose Planning Horizon Case

Step 7. Construct $s\left(0, \left(1 + \frac{c_x^+}{c_x^-}\right), \frac{c_u^1}{U_1}\right)$ and check if

$$\sum_{j=1}^{\left|s\left(0,\left(1+\frac{c_x^+}{c_x^-}\right),\frac{c_u^1}{U_1}\right)\right|} \sum_{n=1}^{N} U_n u_n(t_j)(t_{j+1} - t_j) > D$$

then set $\bar{\psi} = \frac{c_u^1}{U_1}$. Otherwise go to Step 13.

Step 8. Set $\bar{t}_1' = (1 + \frac{c_x^+}{c_x^-}) t_d + \frac{c_u^1}{c_x^+ U_1} - T \frac{c_x^-}{c_x^+}$ and $\bar{t}_2' = -\frac{c_x^+}{c_x^-}(\bar{t}_1 - t_d) + t_d$.

Step 9. Define switching points by (26) and (27). Construct $s(\bar{t}_1', \bar{t}_2', \overline{\psi})$, $Z(j)$, $Q(j)$, $Q^0(j)$ and $Q^1(j)$, $j = 1, 2, \ldots, |s(\bar{t}_1', \bar{t}_2', \overline{\psi})|$. Determine $u_n(t_j)$ by (29) for $j = 1, 2, \ldots, |s(\bar{t}_1', \bar{t}_2', \overline{\psi})|$ and $n = 1, \ldots, N$.

Step 10. Find the shortest time interval $\tau = \min_j\{t_{j+1} - t_j\}$ and time points $\bar{t}_1$ and $\bar{t}_2$ which satisfy the following two equations:

$$\sum_{j=1}^{|s(\bar{t}_1', \bar{t}_2', \overline{\psi})|} \sum_{n=1}^{N} U_n u_n(t_j)(t_{j+1} - t_j) - D = 0, \quad \bar{t}_2 = -\frac{c_x^+}{c_x^-}(\bar{t}_1 - t_d) + t_d,$$

where

$$t_j = \bar{t}_1 + \frac{c_u^{Q^0(j)}}{c_x^+ U_{Q^0(j)}} - \frac{c_n^1}{c_x^+ U_1} \quad \text{if } Q(j) = 0$$

and

$$t_j = \bar{t}_2 - \frac{c_u^{Q^1(j)}}{c_x^- U_{Q^1(j)}} + \frac{c_n^1}{c_x^- U_1} \quad \text{if } Q(j) = 1.$$

Step 11. If $\bar{t}_1 \in \tau$, then go to Step 12. Otherwise assign $\bar{t}_1' := \bar{t}_1' + \tau$ and $\bar{t}_2' = -\frac{c_x^+}{c_x^-}(\bar{t}_1 - t_d) + t_d$. Go to Step 9.

Step 12. Set the optimal solution as: $u_n^*(t) = 0$ for $0 \leqslant t < \bar{t}_1$ and $\bar{t}_2 \leqslant t \leqslant T$, $u_n^*(t) = u_n(t_j)$ for $t_j \leqslant t < t_{j+1}$, $j = 1, 2, \ldots, |s(\bar{t}_1, \bar{t}_2, \overline{\psi})| - 1$, $n = 1, \ldots, N$. STOP.

Step 13. Set $\bar{t}_1 = 0$, $t' = 0$, $\psi(0) = \frac{c_u^1}{U_1}$, $\overline{\psi}\prime = \psi(0)$, $\bar{t}_2 = \frac{\psi(0)}{c_x^-} + \left(1 + \frac{c_x^+}{c_x^-}\right) t_d - \frac{c_u^1}{c_x^- U_1}$ and $\psi_1 = \psi(0)$.

Step 14. Define switching points by (26) and (27). Construct $s(\bar{t}_1, \bar{t}_2', \overline{\psi}')$, $Z(j)$, $Q(j)$, $Q^0(j)$ and $Q^1(j)$, $j = 1, 2, \ldots, |s(\bar{t}_1', \bar{t}_2', \overline{\psi}')|$. Determine $u_n(t_j)$ by (29) for $j = 1, 2, \ldots, |s(\bar{t}_1', \bar{t}_2', \overline{\psi}')|$ and $n = 1, \ldots, N$.

Step 15. Find the shortest time interval $\tau = \min_j\{t_{j+1} - t_j\}$ and the corresponding co-state interval $\psi = (\psi_1, \psi_2)$, where $\psi_2 = \psi_1 + c_x^+ \tau$ if $t' < t_d$ and $\psi_2 = \psi_1 - c_x^- \tau$ if otherwise. Find the unknowns $\psi(0)$ and $\bar{t}_2$ which satisfy the following two equations:

$$\sum_{j=1}^{|s(0, \bar{t}_2', \psi'(0))|} \sum_{n=1}^{N} U_n u_n(t_j)(t_{j+1} - t_j) - D = 0$$

and

$$\bar{t}_2 = \frac{\psi(0)}{c_x^-} + \left(1 + \frac{c_x^+}{c_x^-}\right) t_d - \frac{c_u^1}{c_x^- U_1},$$

where

$$t_1 = 0, \quad t_j = \frac{c_u^{Q^0(j)}}{c_x^+ U_{Q^0(j)}} - \frac{\psi(0)}{c_x^+} \quad \text{for } j > 1 \quad \text{if } Q(j) = 0$$

and

$$t_j = \bar{t}_2 - \frac{c_u^{Q^1(j)}}{c_x^- U_{Q^1(j)}} + \frac{c_n^1}{c_x^- U_1}.$$

Step 16. If $\psi(0) \in \psi$, then go to Step 12. Otherwise assign $\psi_1 := \psi_2$, $\overline{\psi'} := \psi_2$, $t' := t' + \tau$ and

$$\bar{t}_2' = \frac{\psi_1}{c_x^-} + \left(1 + \frac{c_x^+}{c_x^-}\right) t_d - \frac{c_u^1}{c_x^- U_1}.$$

Go to Step 14.

Theorem 1. *Problem (1)–(6) is solvable in* $O((2N+K)^2\max\{N, \log(2N+K)\})$ *time.*

Proof. First note, the algorithm constructs a solution based on the greedy production rule and Lemmas 1–5 thereby satisfying all optimality conditions derived from the maximum principle. Moreover, due to the fact that constraints (1)–(4) are linear and objective function (6) is piece-wise linear, problem (1)–(6) is unimodal. Thus, the maximum principle results in not only necessary but also sufficient conditions of optimality.

To prove the complexity of the algorithm, we assess it step by step. Step 1 uses the priority rule to sort N machines which requires $O(N\log N)$ operations. Step 2 employs Lemma 2 to verify in O(1) time the case when no production is the optimal solution. If production is profitable, the algorithm proceeds to verify the next case which is due to the pressing horizon (Lemma 3). This is accomplished by subdividing the case of the pressing horizon into two subcases at Steps 3–5. According to Lemma 4, construction of $s(\bar{t}_1, \bar{t}_2\overline{\psi})$ at Step 4 requires sorting of at most $2N+K$ time points thereby resulting in $O((2N+K\log(2N+K))$ complexity. Application of the greedy production rule (29) at Step 5 evidently needs at most $N(2N+K)$ operations. Thus, the first subcase which is characterized by $\bar{t}_1 > 0$ is solved in $O((2N+K)\max\{N, \log(2N+K)\})$ time. If however, $\bar{t}_1 = 0$, the second subcase of the pressing horizon is verified at Step 6. Since Step 6 is of O(1) complexity, the total complexity remains the same. If neither of the two subcases (Steps 3–6) results in a feasible solution, the algorithm proceeds to Step 7 to consider loose production conditions as defined by Lemma 5. Solving the problem under these conditions involves the same two subcaes of $O((2N+K)\max\{N, \log(2N+K)\})$ complexity. The difference is that the unknowns $\bar{t}_1 > 0$ (the first subcase) and $\psi(0)$ (the second subcase) depend on the sequence of the switching points (26) and (27). Therefore, at Steps 9–11 (the first subcase) and Steps 14–16 (the second subcase), the algorithm identifies such subintervals that the sequence of the switching points does not change. These steps are then repeated at most $2N+K$ times to solve the problem for each such subinterval until a feasible and, thus, optimal solution is found. Therefore, the total complexity is $O((2N+K^2\max\{N, \log(2N+K)\})$, as stated in the theorem. □

Corollary 2. *Problem (1)–(6) is solvable in* $O(N)$ *time, if* $T - t_d \leqslant \frac{c_u^1}{c_x^- U_1}$.

Proof. The proof immediately follows from the proof of Theorem 1 for Steps 1 and 2 of the algorithm by taking into account that finding a single machine, $i = \arg\min_n \frac{c_u^n}{U_n}$, is possible without sorting. □

Corollary 3. *Problem (1)–(6) is solvable in* $O((2N+K)\max\{N, \log(2N+K)\})$ *time, if* $X(T) < D$.

Proof. The proof immediately follows from the proof of Theorem 1 for Steps 2–6 of the algorithm. □

Corollaries 2 and 3 present two special cases of problem (1)–(6). The first case is due to the fact that the unit production cost per maximal production rate is greater than the unit backlog cost accumulated from the due date up to the end of the planning horizon: $c_x^-(T - t_d) \leqslant \frac{c_u^1}{U_1}$. The second case is due to the pressing horizon which causes the demand to be unsatisfied by the end of the planning horizon, that is $X(T) < D$. In contrast to the condition of Corollary 2, condition $X(T) < D$, which is adopted in Corollary 3, cannot be verified a priori. This implies that we should solve the problem as though it is given $X(T) < D$ and then simply check whether the demand is indeed unsatisfied (the horizon is pressing) as it is evaluated in the algorithm. This disadvantage, however, has no influence on the total complexity of the algorithm and consequently on tractability of the problem. Indeed, if the production conditions are pressing then we concurrently both check and derive the solution in strongly polynomial time. Secondly, according to Corollary 3, the search for the optimal solution under pressing production conditions, which can be viewed as an a priori condition, is faster than for the optimal solution under loose planning horizon conditions.

Finally, according to Theorem 1, problem (1)–(6) is solvable in $O((2N)^2 \max\{N, \log(2N)\})$ if the maximal total processing rate is constant, i.e., $K = 1$ and

$$R(t) = r \quad \text{for } 0 \leqslant t \leqslant T. \tag{36}$$

Moreover, $K = 1$ implies that construction of $s(\bar{t}_1, \bar{t}_2 \bar{\psi})$ does not involve time points (28) and, thus, sorting at Steps 4, 9, and 14 of the algorithm is no longer needed. That is, the true complexity of the algorithm reduces to $O(4N^3)$ under loose horizon conditions and $O(2N^2)$ under pressing horizon conditions.

6. Example

Consider the same juice blending example with $c_x^+ = 0.01$ and

$$c_x^- = 0.02 \frac{\$}{1 \text{ hour}}, \quad R(t) = \begin{cases} 4 & \text{for } 0 \leqslant t < 80, \\ 4.5 & \text{for } 80 \leqslant t \leqslant 120, \end{cases}$$

$T = 200$ hours, $D = 30\,000$ l, $t_d = 150$ hours and blending rates, costs and priorities shown in Table 1.

Based on the priority rule Table 1 identifies the following optimal sequencing of the blenders, 1-3-4-5-2 (Step 1 of the algorithm). With respect to Lemma 2, we find that $T - t_d = 50 > \frac{c_u^1}{c_x^- U_1} = 2.5$, that is, no production at all cannot be optimal for this system (Step 2). Next we verify the first condition of Lemma 3 (Step 3). Since

$$t_d - \frac{c_x^-}{c_x^+}(T - t_d) = 52.5 \geqslant -\frac{c_u^1}{c_x^+ U_1} = -2.5,$$

then

$$\bar{t}_1 = t_d - \frac{c_x^-}{c_x^+}(T - t_d) + \frac{c_u^1}{c_x^+ U_1} = 55 \text{ hours} \quad \text{and} \quad \bar{t}_2 = T - \frac{c_u^1}{c_x^- U_1} = 197.5 \text{ hours}.$$

This implies that the system produces from $t = 55$ to $t = 197$ and it is idle for $0 \leqslant t < 55$ and $197 \leqslant t \leqslant 200$. Based on the blender priorities, 1-3-4-5-2, and Lemma 3, Steps 4 and 5 of the algorithm result in the following switching points (before the due date from (26)),

$$\frown{t}_1 = 55.71429, \quad \frown{t}_2 = 58.33333, \quad \frown{t}_3 = 58.88889, \quad \text{and} \quad \frown{t}_4 = 60.00 \text{ hours}$$

and (after the due date from (27)),

$$\smile{t}_1 = 197.1429, \quad \smile{t}_2 = 195.8333, \quad \smile{t}_3 = 195.5556, \quad \text{and} \quad \smile{t}_4 = 195.00 \text{ hours}.$$

Thus, $s(\bar{t}_1, \bar{t}_2, \bar{\psi}) = \{55, 55.71429, 58.33333, 58.88889, 60.00, 80, 195.00, 195.5556, 195.8333, 197.1429, 197.5\}$, $Z(1) = 1,\ Z(2) = 2,\ Z(3) = 3,\ Z(4) = 4,\ Z(5) = 5,\ Z(6) = 5,\ Z(7) = 4,\ Z(8) = 3,\ Z(9) = 2,\ Z(10) = 1,$

Table 1
System parameters for the blending example

Blender parameters					
N	1	2	3	4	5
U_n	80	100	70	60	90
c_u^n	4	10	4	5	8
c_u^n/U_n	0.05	0.1	0.057143	0.083333	0.088889
Priority	1	5	2	3	4

$Z(11) = 0$. Therefore by setting $u_n(t) = 0$ for $n = 1,\dots, N$, $0 \leqslant t \leqslant 200$ and employing the greedy production rule we have:

$$
\begin{aligned}
u_1(t) &= 1 \quad \text{for } 55 \leqslant t < 197,\\
u_3(t) &= 1 \quad \text{for } 55.71429 \leqslant t < 197.1429,\\
u_4(t) &= 1 \quad \text{for } 58.3333 \leqslant t < 195.8333,\\
u_5(t) &= 1 \quad \text{for } 58.8889 \leqslant t < 195.5556 \text{ and}\\
u_2(t) &= 0.5 \quad \text{for } 80.00 \leqslant t < 195.00.
\end{aligned}
$$

According to Lemma 3, this solution is optimal (i.e., the horizon is *pressing*) if $X(t) < D = 30000$. Calculating the total production amount for the determined solution (Step 5) we confirm that

$$
\begin{aligned}
X(200) &= \int_{\bar{t}_1}^{\bar{t}_2} \sum_n U_n u_n(t)\mathrm{d}t\\
&= 80(197-55) + 70(197.1429-55.71429) + 60(195.8333-58.3333) + 90(195.556\\
&\quad -58.8889) + 100 * 0.5(195.00-80.00)\\
&= 24800 < 30000.
\end{aligned}
$$

Note, if we replace the given demand with a value which is less than 24800 l, then the conditions of Lemma 3 will not be met. This implies that the production conditions become *loose* and Lemma 5 should be employed (Step 7). Specifically, let $D = 24000$ l, then solving (34) results in $\bar{t}_1 = 58.40426$ and $\bar{t}_2 = 195.7979$ hours (Steps 9–11). Consequently, the algorithm recalculates the remaining switching points

$$
\begin{aligned}
&\frown{t}_1 = 59.11854, \quad \frown{t}_2 = 61.73759, \quad \frown{t}_3 = 62.29314, \quad \text{and} \quad \frown{t}_4 = 63.40426 \text{ (before the due date)},\\
&\smile{t}_1 = 195.4407, \quad \smile{t}_2 = 194.1312, \quad \smile{t}_3 = 193.8534, \quad \text{and} \quad \smile{t}_4 = 193.2979 \text{ (after the due date)}.
\end{aligned}
$$

Therefore, $X(200) = 24000$ l and the optimal greedy production in this case is

$$
\begin{aligned}
u_1(t) &= 1 \quad \text{for } 58.40426 \leqslant t < 195.7979,\\
u_3(t) &= 1 \quad \text{for } 59.11854 \leqslant t < 195.4407,\\
u_4(t) &= 1 \quad \text{for } 61.73759 \leqslant t < 194.1312,\\
u_5(t) &= 1 \quad \text{for } 62.29314 \leqslant t < 193.8534 \text{ and}\\
u_2(t) &= 0.5 \quad \text{for } 80.00 \leqslant t < 193.2979.
\end{aligned}
$$

7. Conclusion

N-machine, renewable resource constrained production scheduling with a due date is formalized and studied as a continuous-time dynamic problem. The objective is to minimize inventory, backlog and production costs. With the aid of the maximum principle, analytical rules for selecting optimal production rates, their switching and sequencing are derived. Given that the resource availability is piece-wise constant over K intervals of time, based on these rules, a polynomial-time scheduling algorithm is developed which solves the problem in $O((2N + K)^2\max\{N, \log(2N + K)\})$ time. Moreover, it is proven that if the planning horizon is so pressing that the demand is unsatiated by the end of the horizon or the level of the resource available is constant in time, the problem is solvable to optimality in $O((2N + K)\max\{N, \log(2N + K)\})$ and $O(4N^3)$ time respectively. We believe more solvable cases can be found with this approach. In addition, the properties of the optimal schedules that were identified in this research could be extended and used in future research for solving generic problems.

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