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Production under periodic demand update prior to a single selling season: A decomposition approach

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Abstract

This paper focuses on dynamic, continuous-time production control problems in the fashion industry. Similar to the classical news-vendor problem, we consider a single product-type and the cumulative demand for items is not known until the end of the production horizon and therefore must be forecasted. Since there are periodic updates before a single selling season, actual demand during a period of time can only be determined by the end of the period. If the overall demand is overestimated, excessive inventory holding and production costs are paid and surpluses are sold at low prices at the end of the production horizon. If it is under-estimated, then sales are lost. The objective is to dynamically determine production orders which minimize overall expected costs. Since the optimal feedback for such a problem is characterized by thresholds evolving with time and system states, there is a significant computational burden in determining them. With the aid of the variational analysis and a decomposition, we derive a closed-form solution for the thresholds. A numerical study carried out to compare the decomposition and straightforward simulation-based solutions indicates the high accuracy of the suggested approach while the computational burden is dramatically reduced.

Keywords: Production control; Systems dynamics

1. Introduction

Inventory management is a key business function for companies operating with inventories that may quickly become obsolete, spoil, or have a future that is uncertain beyond a single period. This paper is motivated by the problem arising in fashion industry or in companies, which supply various garment accessories for production of fashion industry goods. The demand for these accessory items is unknown prior to a selling season. Once the season starts it is too late to produce, since fashion good manufacturers cannot halt their production to wait for deliveries. To prevent loss of sales and clients, accessory manufacturers tend to keep large stocks of end items. This, however, ties up sizeable amounts of capital.

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Forecasting demand is an efficient way of reducing uncertainty and excessive inventories and manufacturers will expend much time and effort at professional exhibitions of leading fashion designers in attempt to foresee future trends and accessory needs in the upcoming season. In addition catalogs and samples are periodically sent to fashion good manufacturers in order to update demand forecasts and get advance orders.

A typical approach to incorporating demand uncertainty is to assume a random but known and stationary distribution for the demand in each demand period. Porteus (1990) gives examples of this approach which commonly results in base stock models. In practice, these models are applied, for example, to fashion or seasonal products by large international apparel brands when the products are characterized by the supply lead-time comparable to the length of the selling period (Fuloria and Wadhwa, 1995). In this paper, since we focus on this type of seasonal goods, we assume that demand is independent (stationary) across time, as is the case in many other studies devoted to seasonal goods (e.g., Bitran et al., 1997; Federgruen and Heching, 1999; Feng and Gallego, 1995; Gallego and van Ryzin, 1994).

The problem under discussion fits the well-known class of single-period inventory models, which are frequently referred to as news vendor or newsboy problems. An extensive literature review on various extensions of the classical newsboy problem and related inventory control models can be found in Khouja (1999) and Silver et al. (1998). Although the importance of extensions to models with more than one period to prepare for the selling season has been stressed in this literature, only problems with one additional order or urgent reorder have been solved (see, for example, Veinott, 1966 and Wright, 1969). Lau and Lau (1998) considering a single-period newsboy type product which can be ordered twice during a period, show that the decision is substantially more complicated than for the simple one-order-per-period newsboy problem typically solvable with a base stock policy (ordering up to a certain level). They suggest several heuristic decision rules. Murray and Silver (1966), Hausman and Peterson (1972), Bitran et al. (1986) and Matsuo (1990) consider a number of sub-periods to prepare for the selling season. These models commonly utilize special product and demand parameters to optimize operations over each sub-period. This results in either a stochastic mixed-integer or dynamic programming. Since both outcomes pose significant computational problems, heuristics are commonly suggested. The heuristics provide various computational shortcuts based on (i) limiting the ability to adjust production in response to demand updates, (ii) reducing a multi-period problem to a single period problem or to a number of simplified problems (multi-phase heuristics), and (iii) replacing stochastic programming with a deterministic integer programming.

In contrast to these periodic review models with finite horizons, there is a stream of research studies on the use of base stock policies with advance demand information (BSADI) for continuous review of production/inventory systems operating over an infinite planning horizon. Policies of this type have been investigated in a number of papers (see, for example, Hariharan and Zipkin, 1995; Toktay and Wein, 2001; Karaesmen et al., 2004; Wijngaard, 2004). Ozer and Wei (2004) address periodic review, capacitated, finite and infinite horizon production faced by a manufacturer who has the ability to obtain advance demand information. They show that for such production systems, even when fixed costs are zero, base stock levels or thresholds evolve over time with a system's state. This radically affects the computational burden for a straightforward, backward induction algorithm that they use to numerically solve the problem. Consequently, developing simple-to-calculate, closed forms for the thresholds to provide a good approximation of the optimal solution is a challenging contribution to both engineering and operations research literature.

In this paper we deal with a capacitated system operating over a finite planning horizon. Similar to the above newsboy type papers, we consider inventory costs incurred only by the end of the planning horizon (single-review). In contrast to what appears in the literature, this paper derives a closed form solution for thresholds evolving over a continuous-time finite horizon under periodic demand updates. As a result, heavy numerical computations of the thresholds can be avoided. The derivation of the closed form solutions is accomplished with a decomposition method. The general problem presented in Section 2 is decomposed into two sub-problems. First, a lower bound is derived by minimizing the expected cost over all possible realizations without imposing the non-anticipativity condition (Section 3). This implies that the control which could provide such a cost on-line does not always exist. Then we consider an on-line control at a time point; impose non-anticipativity (which increases the expected cost found at the first step) at this time point; and apply a small control variation to minimize the change in the cost function. The minimization results in a feedback policy (Section 4). Section 5 presents an example. As shown in our simulation results (Section 6), this lower bound-guided solution method provides a very good approximation to the optimal solution. Section 7 summarizes the results.

2. The model

Consider a manufacturing system producing a single product-type to satisfy a cumulative demand, d, for the product-type by the end of a production horizon, T. The production horizon T is subdivided into K with not necessarily equal length periods, defined by points t_k, $k = 1,\ldots,K$, $t_0 = 0$, $t_K = T$. The demand for a period k, d_k is a random parameter for which realization, D_k, is known (updated), only at the end of the period. As discussed in the introduction, we assume that d_k are independent parameters characterized by the probability density function $f_k(D_k)$ and cumulative distribution $F_k(D_k)$ with mean μ_k and standard deviation σ_k. The difference between the inventory level $X(t_k)$ and the updated demand D_k is assessed at the end of period k and the production load $u(t)$ of the system is decided for the next time period $t_k \leqslant t < t_{k+1}$. This implies that the control function $u(t)$ generally depends on realization $\xi = \langle D_1,\ldots,d_k\rangle$, $u(t,\xi)$.

Denote the set of all possible realizations $\{\xi\}$ over the entire production horizon as R and the set of all realizations ξ, $\xi \in R$, which coincide with a realization from the beginning of the production horizon through time period k, as $R(k,\xi)$. That is,

$$R(k,\xi) = \{\xi' = \langle D'_1,\ldots,D'_K\rangle | \xi' \in R \quad \text{and} \quad D'_j = D_j, \text{ for } 1 \leqslant j \leqslant k\}. \tag{1}$$

Thus, $R(k,\xi)$, consists of only those realizations that still can happen in future if by time t_k we observe the realization ξ. Therefore, the control is feasible if in addition to the boundary loading constraint:

$$0 \leqslant u(t,\xi) \leqslant 1, \quad \xi \in R, \quad 0 \leqslant t \leqslant T, \tag{2}$$

we have non-anticipativity

$$u(t,\xi) = u(t,\xi') \quad \text{for all} \quad \xi' \in R(k,\xi),\ 0 \leqslant t < t_{k+1}. \tag{3}$$

Then the inventory level, $X(t)$, by time t for realization ξ, $X(t,\xi)$ is described by the following equation:

$$\dot{X}(t,\xi) = Uu(t,\xi), \quad X(0,\xi) = X(0), \tag{4}$$

where U is the maximum production rate.

There are operational costs associated with the production process (4). Specifically, at each t, the system incurs a production cost, $cu(t,\xi)$ and an inventory holding cost, $hX(t,\xi)$. Penalties are paid for backlogs (under-production) at the end of the production horizon T, $p^-(X(T,\xi) - \sum_{k=1}^K D_k)$, when $X(T,\xi) - \sum_{k=1}^K D_k < 0$ and for overproduction, $p^+(\sum_{k=1}^K D_k - X(T,\xi))$, when $X(T,\xi) - \sum_{k=1}^K D_k > 0$.

Our goal is to determine production loads $u(t,\xi)$ for each realization ξ, $\xi \in R$, so that the expected total cost, J, is minimized over the production horizon:

$$J = E_R\left[\int_0^T [hX(t,\xi) + cu(t,\xi)]\mathrm{d}t + p^- \max\left\{0, \sum_{k=1}^K D_k - X(T,\xi)\right\} + p^+ \max\left\{0, X(T,\xi) - \sum_{k=1}^K D_k\right\}\right] \to \min, \tag{5}$$

where $E_\mathrm{R}[\cdot]$ is the expectation operator taken over all realizations R. From (5) we observe that in order to derive a lower bound by minimizing the expected cost over all possible realizations R without imposing the non-anticipativity condition (3), we need to identify a deterministic relationship between each realization ξ, cumulative production demand $\sum_{j=1}^K D_j$ and optimal production load $u(t,\xi)$, $t_{k-1} \leqslant t \leqslant T$. This relationship is studied in Section 3.

3. The relationship between optimal production loads and cumulative demands

To derive the relationship between the optimal production load and demand, we assume that loading $u(t,\xi)$ has been made up to point t_{k-1}, and $\sum_{j=1}^K D_j$ is known. Then problem (1)–(5) takes the following deterministic form:

$$\int_{t_{k-1}}^{T}[hX(t,\xi)+cu(t,\xi)]\mathrm{d}t+p^{-}\max\left\{0,\sum_{j=1}^{K}D_j-X(T,\xi)\right\}+p^{+}\max\left\{0,-\sum_{j=1}^{K}D_j+X(T,\xi)\right\}\rightarrow\min \tag{6}$$

subject to

$$X(t,\xi)=X(t_{k-1},\xi)+\int_{t_{k-1}}^{t}Uu(s,\xi)\mathrm{d}s; \tag{7}$$

$$0\leqslant u(t,\xi)\leqslant 1,\ \ t_{k-1}\leqslant t\leqslant T. \tag{8}$$

Applying the maximum principle (see, for example, Maimon et al., 1998), we construct the Hamiltonian:

$$H(t)=-cu(t,\xi)-hX(t,\xi)+\psi(t)Uu(t,\xi), \tag{9}$$

where the multiplier ψ (t) is referred to as a costate variable and satisfies the following costate equation:

$$\dot{\psi}(t)=h \tag{10}$$

with transversality (boundary) constraint:

$$\psi(T)=\begin{cases}p^{-}, & \text{if}\quad \sum_{j=1}^{K}D_j>X(T,\xi);\\ -p^{+}, & \text{if}\quad \sum_{j=1}^{K}D_j<X(T,\xi);\\ p\in[-p^{+},p^{-}], & \text{if}\quad \sum_{j=1}^{K}D_j=X(T,\xi).\end{cases} \tag{11}$$

According to the maximum principle, the Hamiltonian is maximized for each t by the optimal controls $u(t,\xi)$. Therefore, by considering only control-dependent terms of the Hamiltonian we obtain:

$$u(t,\xi)=\begin{cases}1,\text{if }\ \psi(t)>\frac{c}{U};\\ w\in[0,1],\text{if }\ \psi(t)=\frac{c}{U};\\ 0,\text{if }\ \psi(t)<\frac{c}{U}.\end{cases} \tag{12}$$

Thus under the optimal solution, the system can be idle $\left(\psi(t)<\frac{c}{U}\right)$; work with maximum load $\left(\psi(t)>\frac{c}{U}\right)$; or enter the singular regime $\left(\psi(t)=\frac{c}{U}\right)$ which is characterized by an intermediate load between 0 and 1. Furthermore, by differentiating $\psi(t)=\frac{c}{U}$ over an interval of time, we find that $\dot{\psi}(t)=0$, which contradicts Eq. (10), i.e., the singular regime cannot ever occur over an interval of time and, thus, condition (12) simplifies to:

$$u(t,\xi)=\begin{cases}1, & \text{if }\ \psi(t)>\frac{c}{U};\\ 0, & \text{if }\ \psi(t)\leqslant\frac{c}{U}.\end{cases} \tag{13}$$

Consequently, a straightforward non-production condition is as follows.

Lemma 1. *Given problem (6)–(8) with inventory level at the end of a period, $k-1$, $X(t_{k-1},\xi)$ and cumulative demand $\sum_{j=1}^{K}D_j$, if $X(t_{k-1},\xi)\geqslant\sum_{j=1}^{K}D_j$, then it is optimal not to produce, $u(t,\xi)=0$ for $t_{k-1}\leqslant t\leqslant T$.*

Proof. The proof is immediate as there is no longer sense to produce if $X(t_{k-1},\xi)\geqslant\sum_{j=1}^{K}D_j$. □

We next distinguish between two production systems: (a) a system with high unit under-production penalties relative to the unit production cost per time unit plus unit holding cost over the entire production horizon, i.e.,

$$p^{-}\geqslant\frac{c}{U}+hT \tag{14}$$

and (b) a system with moderate under-production penalties, $\frac{c}{U}<p^{-}<\frac{c}{U}+hT$. Note, that if under-production penalties are so low that $p^{-}\leqslant\frac{c}{U}$, then production is not profitable at all. The next two lemmas derive an optimal solution for the system with high penalties for under-production.

Lemma 2. *Given problem (6)–(8) with inventory level at the end of a period,* $k-1$, $X(t_{k-1},\xi)$, *cumulative demand* $\sum_{j=1}^{K}D_j$ *and* $p^- \geqslant \frac{c}{U}+hT$, *if* $X(t_{k-1},\xi) < \sum_{j=1}^{K}D_j \leqslant X(t_{k-1},\xi)+U(T-t_{k-1})$, *then the optimal production load is determined by*

$$u(t,\xi)=\begin{cases}0, & \text{for } t_{k-1}\leqslant t<t^* \\ 1, & \text{for } t^*\leqslant t\leqslant T\end{cases}, \tag{15}$$

where

$$t^* = T-\frac{\sum_{j=1}^{K}D_j - X(t_{k-1},\xi)}{U}. \tag{16}$$

Proof. The proof is presented in the Appendix. □

Note, that if we relax the high under-production penalty condition $p^- \geqslant \frac{c}{U}+hT$ to a moderate level $\frac{c}{U}<p^-<\frac{c}{U}+hT$, then the optimal solution determined by Lemma 2 will remain the same if inequality

$$\frac{c}{U}+h(T-t^*)\leqslant p^- \tag{17}$$

still holds, i.e., $t^*\geqslant t'$, where

$$t'=\frac{c}{hU}+T-\frac{p^-}{h}. \tag{18}$$

Lemma 3. *Given problem (6)–(8) with inventory level at the end of a period,* $k-1$, $X(t_{k-1},\xi)$, *cumulative demand* $\sum_{j=1}^{K}D_j$ *and* $p^- \geqslant \frac{c}{U}+hT$, *if* $\sum_{j=1}^{K}D_j > X(t_{k-1},\xi)+U(T-t_{k-1})$, *then the maximum load is optimal over the remaining production horizon,* $u(t,\xi)=1$ *for* $t_{k-1}\leqslant t\leqslant T$.

Proof. The proof is contained in the Appendix. □

Similar to Lemma 2, we note that condition

$$-h(T-t_{k-1})+p^- > \frac{c}{U} \tag{19}$$

holds for a moderate under-production penalty system if $t'\leqslant t_{k-1}$.

Two important facts follow from Lemmas 1–3. First, the relationships between the optimal production load and a demand realization, ξ, are very similar for both high and moderate penalty systems. Moreover, any moderate penalty system becomes a high penalty one starting from a point, t_k. Therefore, to avoid excessive mathematical expressions, we further focus on a high under-production penalty system. The second fact which immediately follows from Lemmas 1–3 is that if the anticipativity constraint is relaxed, the optimal production load depends on the current inventory level and cumulative demand $\sum_{j=1}^{K}D_j$, rather than its sequence. Therefore to derive a lower bound of (5), we can redefine ξ as $\xi=\sum_{j=1}^{K}D_j$. Let $\xi'=\sum_{j=1}^{K}D'_j$, then conditions (1) and (3) can now be replaced with

$$R(k,\xi)=\left\{\text{all } \xi'\in R \text{ such that } \quad \sum_{j=1}^{k}D'_j=\sum_{j=1}^{k}D_j\right\}, \tag{20}$$

$$X(t,\xi)=X(t,\xi'),\quad \xi'\in R(k,\xi),\quad t_k\leqslant t<t_{k+1}, \tag{21}$$

respectively.

4. Feedback policy

Given inventory level at the end of a period, $k-1$, $X(t_{k-1},\xi)$ and realized cumulative demand $\sum_{j=1}^{k-1}D_j$, based on the optimal relationship derived in Lemmas 1–3 and conditions (20) and (21), the expectation operator in objective function (5) results in

$$
\begin{aligned}
J^L = {} & E_{R(k-1),\xi}\left[\int_0^T [hX(\tau,\xi) + cu(\tau,\xi)]\mathrm{d}t\right] \\
& - \int_{X(t_{k-1},\xi)+U(T-t_{k-1})-\sum_{j-1}^{k-1} D_j}^{\infty} p^-\left(X(t_{k-1},\xi) + U(T-t_{k-1}) - w - \sum_{j=1}^{k-1} D_j\right)\phi_k(w)\mathrm{d}w \\
& + \int_{X(t_{k-1},\xi)-\sum_{j-1}^{k-1} D_j}^{X(t_{k-1},\xi)+U(T-t_{k-1})-\sum_{j-1}^{k-1} D_j} p\left(X(t_{k-1},\xi) + U(T-t^*) - w - \sum_{j=1}^{k-1} D_j\right)\phi_k(w)\mathrm{d}w \\
& + \int_{-\infty}^{X(t_{k-1},\xi)-\sum_{j-1}^{k-1} D_j} p\left(X(t_{k-1},\xi) - w - \sum_{j=1}^{k-1} D_j\right)\phi_k(w)\mathrm{d}w,
\end{aligned}
\tag{22}
$$

where $w = \sum_{j=k}^{K} D'_j$, $\phi_k(w)$ is the probability density function obtained as a convolution of the probability density functions $f_j(D_j)$ over periods $j = k$ to $j = K$. If, for example, the popular normal distribution is employed to describe demands, then $\phi_k(w)$ is normal as well. The cumulative function of $\phi_k(w)$ is denoted by $\Phi_k(w)$. The mean and the standard deviation of these functions are determined straightforwardly as, $\mu_k + \mu_{k+1} + \cdots + \mu_K$, and $\sqrt{\sigma_k^2 + \sigma_2^2 + \cdots + \sigma_K^2}$, respectively.

Expression (22) is a lower bound of the objective function (5), since it is obtained by minimizing the expected cost over all possible realizations without imposing the non-anticipativity condition. This implies that the control which could provide such a cost on-line does not always exist.

We next consider an on-line control at a time point t, $u(t,\xi)$ and impose non-anticipativity (which increases the lower bound (22)) at this time point. Similar to the deterministic maximum principle, we derive optimality conditions by considering a small variation of the optimal control $\delta u(t,\xi)$ to minimize the change in the lower bound.

Given update information at the end of a period, $k-1$, D_{k-1} (and, thus, $\sum_{j=1}^{k-1} D_j$), inventory level $X(t_{k-1},\xi)$ and a control, $u(t,\xi)$, $t_{k-1} \leqslant t < t_k$, we apply a needle, ε, control variation for the realization ξ at point t:

$$
\delta u(\tau,\xi) = \begin{cases} \delta u, & \text{if } t \leqslant \tau \leqslant t + \varepsilon, \\ 0, & \text{otherwise.} \end{cases}
\tag{23}
$$

Then according to (3) we have $u(\tau,\xi) = u(\tau,\xi')$ for all $\xi' \in R(k-1,\xi)$, $t_{k-1} \leqslant \tau < t_k$, and therefore $\delta u(\tau,\xi) = \delta u(\tau,\xi')$ for all $\xi' \in R(k-1,\xi)$, $t_{k-1} \leqslant \tau < t_k$. This, by taking into account (23), results in

$$
\delta u(\tau,\xi) = \delta u(\tau,\xi') \text{ for all } \xi' \in R(k-1,\xi),\ 0 \leqslant \tau \leqslant T.
\tag{24}
$$

Consequently, the influence of variation (23) and (24) on the inventory level $X(t,\xi)$ is

$$
\delta X(\tau,\xi') = \begin{cases} \varepsilon U \delta u, & \text{if } \tau > t \\ 0, & \text{otherwise} \end{cases} \text{for all } \xi' \in R(k-1,\xi).
\tag{25}
$$

The optimality conditions derived by the variation for problem (1)–(4) and (22) are summarized in the following lemma.

Lemma 4. *Given inventory level at the end of a period, $k-1$, $X(t_{k-1},\xi)$ and cumulative demand update $\sum_{j=1}^{k-1} D_j$, the optimal control for $t_{k-1} \leqslant t < t_k$ is determined by*

$$
u(t,\xi) = 1, \text{ if } \varphi_k(t,\xi) > \frac{c}{U}, \quad u(t,\xi) = 0, \text{ if } \varphi_k(t,\xi) \leqslant \frac{c}{U},
\tag{26}
$$

where $\varphi_k(t,\xi) = -h(T-t) + p^- - (p^+ + p^-)\Phi_k\left(X(t_{k-1},\xi) + U(T-t_k) + \int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1} D_j\right) + p^+ \int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau\phi_k\left(X(t_{k-1},\xi) + \int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1} D_j\right)$ *for* $k < K$ *and* $t_{k-1} \leqslant t \leqslant T$ *and* $\varphi_k(t,\xi) = -h(T-t) + p^- - (p^+ + p^-)\Phi_k\left(X(t_{k-1},\xi) + \int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1} D_j\right)$ *for* $k = K$ *and* $t_{k-1} \leqslant t \leqslant T$.

Proof. The proof is presented in the Appendix. □

Based on the optimality conditions derived in Lemma 4, we now can determine feedback control. Similar to the analysis of the deterministic version (6)–(8), we determine three types of solutions in the following three lemmas, which correspond to Lemmas 1–3. Specifically, Lemma 5 derives a condition on the relationship between the current inventory level $X(t_{k-1},\xi)$; latest demand update $\sum_{j=1}^{k-1}D_j$; system costs h, c, p^+, p^-; and the planning horizon T under which non-production is most advantageous along the entire period $t_{k-1}\leqslant t<t_k$.

Lemma 5. *Given inventory level at the end of a period, $k-1$, $X(t_{k-1},\xi)$ and cumulative demand update $\sum_{j=1}^{k-1}D_j$. If $Th+\frac{c}{U}-p^-\geqslant ht_k-(p^++p^-)\Phi_k\Big(X(t_{k-1},\xi)+U(T-t_{k-1})-\sum_{j-1}^{k-1}D_j\Big)$, then condition (26) is satisfied by $u(t,\xi)=0$ for $t_{k-1}\leqslant t<t_k$.*

Proof. Consider the following non-production solution for system (2), (4) and (A4)

$$u(t,\xi)=0 \text{ for } t_{k-1}\leqslant t<t_k;\ X(t_k,\xi)=X(t_{k-1},\xi);\ \varphi_k(T,\xi)=\varphi_k(t_{k-1},\xi)+h(T-t_{k-1}). \tag{27}$$

By setting $\varphi_k(t_{k-1})=-h(T-t_{k-1})-p^-+(p^++p^-)\Phi_k\Big(X(t_{k-1},\xi)+U(T-t_k)-\sum_{j=1}^{k-1}D_j\Big)$, we observe that solution (27) is feasible with respect to (2), (4) and (A4) and meets the idling condition from Lemma 4 for $t_{k-1}\leqslant t<t_k$ if $\varphi_k(t_k)\leqslant\frac{c}{U}$, i.e., if $-h(T-t_k)+p^--(p^++p^-)\Phi_k\Big(X(t_{k-1},\xi)+U(T-t_k)-\sum_{j=1}^{k-1}D_j\Big)\leqslant\frac{c}{U}$, as stated in the lemma. □

In contrast to Lemmas 5 and 6 presents a closed form solution characterized by a breaking point, t^s, so that the production control for period $t_{k-1}\leqslant t<t_k$ changes at this point from no production to the maximum load.

Lemma 6. *Given inventory level at the end of a period, $k-1$, $X(t_{k-1},\xi)$ and cumulative demand update $\sum_{j=1}^{k-1}D_j$. If*

$$\begin{aligned}&t_{k-1}h-(p^++p^-)\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+U(t_k-t_{k-1})-\sum_{j=1}^{k-1}D_j\right)\\&\quad+Ip^+U(t_k-t_{k-1})\phi_k\left(X(t_{k-1},\xi)+U(t_k-t_{k-1})-\sum_{j=1}^{k-1}D_j\right)\\&\quad\leqslant Th+\frac{c}{U}-p^-<ht_k-(p^-+p^+)\Phi_k\left(X(t_{k-1},\xi)+U(t_k-t_k)-\sum_{j=1}^{k-1}D_j\right),\end{aligned}$$

then condition (26) is satisfied by

$$u(t,\xi)=\begin{cases}0, & \text{for } t_{k-1}\leqslant t<t^S,\\ 1, & \text{for } t^S\leqslant t<t_k,\end{cases}$$

where

$$t^s=T+\frac{c}{hU}-\frac{\varphi_k(T,\xi)}{h},\quad I=\begin{cases}0, & \text{if } k=K,\\ 1, & \text{otherwise},\end{cases}$$

and

$$\begin{aligned}\varphi_k(T,\xi)&=p^--(p^++p^-)\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+U(t_k-t^s)-\sum_{j=1}^{k-1}D_j\right)\\&\quad+p^+U(t_k-t^s)\phi_k\left(X(t_{k-1},\xi)+U(t_k-t^s)-\sum_{j=1}^{k-1}D_j\right)\ \text{for } k<K;\\ \varphi_k(T,\xi)&=p^--(p^++p^-)\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+U(t_k-t^s)-\sum_{j=1}^{k-1}D_j\right)\ \text{for } k=K.\end{aligned}$$

Proof. Consider a solution for system (2), (4) and (A4), which consists of no production until a breaking point, t^s, followed by maximum load production:

$$u(t,\xi)=\begin{cases}0, & \text{for } t_{k-1}\leqslant t<t^S\\ 1, & \text{for } t^S\leqslant t<t_k\end{cases},X(t_k,\xi)=X(t_{k-1},\xi)+U(t_k-t^s);\quad \text{and } \varphi_k(T,\xi)=\frac{c}{U}+h(T-t^s). \tag{28}$$

Using (28) we determine

$$t^s=T+\frac{c}{hU}-\frac{\varphi_k(T,\xi)}{h}, \tag{29}$$

where

$$\varphi_k(T,\xi)=p^- -(p^+ +p^-)\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+U(t_k-t^s)-\sum_{j=1}^{k-1}D_j\right) + p^+U(t_k-t^s)\phi_k\left(X(t_{k-1},\xi)+U(t_k-t^s)-\sum_{j=1}^{k-1}D_j\right) \text{ for } k<K.$$

Solution (28) and (29) is feasible and meets (26) for $t_{k-1}\leqslant t<t_k$ if $t_{k-1}\leqslant t^s<t_k$. Using (29) the right-hand condition, $t^s<t_k$, becomes $t^s=T+\frac{c}{hU}-\frac{\varphi_k(T,\xi)}{h}<t_k$, that is,

$$T+\frac{c}{hU}-\frac{p^-}{h}+\left(\frac{p^+ +p^-}{h}\right)\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+U(t_k-t^s)-\sum_{j=1}^{k-1}D_j\right) -\frac{p^+}{h}U(t_k-t^s)\phi_k\left(X(t_{k-1},\xi)+U(t_k-t^s)-\sum_{j=1}^{k-1}D_j\right)<t_k.$$

This condition holds if in the limit case, $t^s=t_k$, we have:

$$t_k>T+\frac{c}{hU}-\frac{p^-}{h}+\left(\frac{p^+ +p^-}{h}\right)\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)-\sum_{j=1}^{k-1}D_j\right),$$

as stated in the lemma. Similarly, the other condition of the lemma is obtained from $t_{k-1}\leqslant t^s$, i.e.,

$$T+\frac{c}{hU}-\frac{p^-}{h}+\left(\frac{p^+ +p^-}{h}\right)\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+U(t_k-t_{k-1})-\sum_{j=1}^{k-1}D_j\right) -\frac{p^+}{h}U(t_k-t^s)\phi_k\left(X(t_{k-1},\xi)+U(t_k-t^s)-\sum_{j=1}^{k-1}D_j\right)\geqslant t_{k-1}. \qquad \square$$

Lemma 7 complements conditions of Lemmas 5 and 6. It provides a relationship between the system parameters under which production with maximum load is beneficial along the entire period $t_{k-1}\leqslant t<t_k$.

Lemma 7. *Given inventory level at the end of a period,* $k-1$, $X(t_{k-1},\xi)$ *and cumulative demand update* $\sum_{j=1}^{k-1}D_j$.

$$\textit{If}\quad Th+\frac{c}{U}-p^-<t_{k-1}h-(p^+ +p^-)\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+U(t_k-t_{k-1})-\sum_{j=1}^{k-1}D_j\right) +Ip^+U(t_k-t_{k-1})\phi_k\left(X(t_{k-1},\xi)+U(t_k-t_{k-1})-\sum_{j=1}^{k-1}D_j\right),$$

then condition (26) *is satisfied by* $u(t,\xi)=1$ *for* $t_{k-1}\leqslant t<t_k$.

Proof. Consider the maximum load solution for system (2), (4) and (A4),

$$u(t,\xi) = U \text{ for } t_{k-1} \le t \leqslant t_k; X(t_k,\xi) = X(t_{k-1},\xi) + U(t_k - t_{k-1}); \quad \text{and}$$
$$\varphi_k(T,\xi) = \frac{c}{U} + h(T - t_{k-1}). \tag{30}$$

As with Lemmas 5 and 6, by substituting (A4), we determine $\psi(t_{k-1})$, which is feasible and meets the full-load condition from Lemma 4 if $\psi(t_{k-1}) > \frac{c}{U}$, which readily results in the condition stated in the lemma. □

Lemmas 5–7 present conditions which involve functions of the current inventory level and demand update. Let $z_k = X(t_{k-1},\xi) - \sum_{j=1}^{k-1} D_j$, $(p^+ + p^-)\phi_k(z_k + U(T - t_k) + U(t_k - t_{k-1})) \geqslant -p^+ U(t_k - t_{k-1}) \mathrm{d}\phi_k(z_k + U(t_k - t_{k-1}))/\mathrm{d}z_k$, threshold a_k satisfy $Th + \frac{c}{U} - p^- = ht_k - (p^+ + p^-)\Phi_k(a_k + U(T - t_k))$, threshold b_k satisfy

$$\begin{aligned} &Th + \frac{c}{U} - p^- = t_{k-1}h(p^+ + p^-)\Phi_k(b_k + U(T - t_k)) - + Ip^+ U(t_k - t_{k-1})\phi_k(b_k + U(t_k - t_{k-1})), \\ &e_k(t) = \begin{cases} 0, & \text{for } t_{k-1} \leqslant t < t^S \\ 1, & \text{for } t^S \leqslant t < t_k \end{cases} \end{aligned} \tag{31}$$

and t^s be determined by (29), where $I = \begin{cases} 0, & \text{if } k = K \\ 1, & \text{otherwise} \end{cases}$. Then the closed form control derived in Lemmas 5–7 can be presented in an explicit threshold form of combined ordinary and integral feedback policy for $t_{k-1} \leqslant t < t_k$, $k = 1,\ldots,K$:

$$u(t,\xi) = \begin{cases} 0, & \text{if } X(t_{k-1},\xi) - \sum_{j=1}^{k-1} D_j \geqslant a_k; \\ e_k(t), & \text{if } b_k \leqslant X(t_{k-1},\xi) - \sum_{j=1}^{k-1} D_j < a_k; \\ 1, & \text{if } X(t_{k-1},\xi) - \sum_{j=1}^{k-1} D_j < b_k. \end{cases} \tag{32}$$

5. Example

Consider a production horizon of six weeks, each week containing five working days, i.e., $T = 30$ working days. The horizon is subdivided into six periods, each of which is five days long. Production demand for each period (week) is assumed to follow normal probability distribution with a mean of 250 product units and a standard deviation of 20 units. The maximum daily production rate is 60 units. Initial inventory $X(0)$ is ten units. Surplus (p^+) and shortage (p^-) costs are \$0.02 and \$0.1 per product unit respectively. Production (c) and product unit inventory holding (h) costs are \$1 and \$0.001 per time unit, respectively.

We denote the left-hand side of the Lemma 6 condition as

$$\begin{aligned} \mathrm{LHS}_k = {} & t_{k-1}h - (p^+ + p^-)\Phi_k\left(X(t_{k-1},\xi) + U(T - t_k) + U(t_k - t_{k-1}) - \sum_{j=1}^{k-1} D_j\right) \\ & + Ip^+ U(t_k - t_{k-1})\phi_k\left(X(t_{k-1},\xi) + U(t_k - t_{k-1}) - \sum_{j=1}^{k-1} D_j\right) \end{aligned}$$

and the right-hand side as

$$\mathrm{RHS}_k = ht_k - (p^+ + p^-)\Phi_k\left(X(t_{k-1},\xi) + U(T - t_k) - \sum_{j=1}^{k-1} D_j\right).$$

The central part of this condition is $Th + \frac{c}{U} - p^- = -0.05333$. Then the optimal control determined by Lemmas 5–7 takes the following form:

Table 1
Computation results for the example

σ_k	μ_k	t^s	LHS_k	RHS_k	$u(t)$	D_{k-1}	$X(t_{k-1})$	t_{k-1}	$\sum_{j=1}^{k-1} D_j$
48.9897	1500	–	−0.12000	−0.0647	0	0	10	0	0
44.7213	1250	5.5178	−0.08485	0.010	$e_2(t)$	230	10	5	230
40	1000	–	−0.00209	0.015	1	300	278.9267	10	530
34.6410	750	–	0.00657	0.020	1	250	578.9267	15	780
28.2842	500	20.431	−0.08161	0.025	$e_5(t)$	170	878.9267	20	950
20	250	25.384	−0.08005	0.030	$e_6(t)$	230	1153.051	25	1180
–	–	–	–	–	–	220	1430.000	30	1400

$$u(t) = \begin{cases} 0, & \text{if } \mathrm{RHS}_k \geqslant 0.05333; \\ e_k(t), & \text{if } \mathrm{LHS}_k \leqslant 0.05333 < \mathrm{RHS}_k; \\ 1, & \text{if } \mathrm{LHS}_k > 0.05333, \end{cases}$$

where $e_k(t)$ is determined by (31). Breaking points, t^s, are found by solving Eq. (29) with the aid of the GOAL SEEK function of EXCEL.

Table 1 implies that $u(t) = 0$ for $0 \leqslant t < 5$ is calculated when there is still no demand update. By the end of this period, the first update results in $D_1 = 230$ units. The control for the second period is $u(t) = 0$ for $5 \leqslant t < 5.5178$, $u(t) = 1$ for $0.5178 \leqslant t < 10$ and it takes into account the update for the previous period. Correspondingly, $u(t) = 1$ for $10 \leqslant t < 20$; $u(t) = 0$ for $20 \leqslant t < 20.431$; and $u(t) = 1$ for $20.431 \leqslant t < 25$. In the last period$u(t) = 0$ for $25 \leqslant t < 25.384$ and $u(t) = 1$ for $20.384 \leqslant t \leqslant 30$, the total inventory level results in 1430 units and the total demand reaches 1400 units.

6. Simulation analysis

In this section we use simulation to assess the suggested decomposition approach. This is accomplished in two different ways.

First, we compare the suggested approach with the optimal solution by determining optimal thresholds a_k^* via a full search so that the cost function (5) is minimized. As in Eq. (32), we use a feedback policy determined by the optimal thresholds,

$$u(t,\xi) = \begin{cases} 0, & \text{if } X(t_{k-1}) - \sum_{j=1}^{k-1} D_j \geqslant a_k^*; \\ e_k(t), & \text{if } a_{k-1}^* - U(t_k - t_{k-1}) \leqslant X(t_{k-1}) - \sum_{j=1}^{k-1} D_j < a_k^*; \quad t_{k-1} \leqslant t < t_k. \\ 1, & \text{if } X(t_{k-1}) - \sum_{j=1}^{k-1} D_j < a_k^* - U(t_k - t_{k-1}), \end{cases} \tag{33}$$

As in Ozer and Wei (2004), this feedback policy triggers at each period a production order $u(t,\xi)$, $t_{k-1} \leqslant t < t_k$ to increase the inventory position (compared to the demand realized) to a level as close as possible to that period's threshold a_k^*, i.e., $e_k(t)$ is defined by (32), and t^S is as follows:

$$t^S = t_k - \frac{a_k^* - \left(X(t_{k-1}) - \sum_{j=1}^{k-1} D_j\right)}{U}. \tag{34}$$

A backward induction algorithm locates optimal thresholds, a_k^*, by simulating various demand realizations first to determine a_{K-1}^* which provides minimum expected cost (5) with policy (33), then to determine a_{K-2}^* and so on. This simulation-based full search induces an enormous computational burden. For example, it takes at least 10 hours to calculate an optimal solution even when there are only two demand updates

($K = 3$). In comparison, the suggested method requires only two minutes at most to calculate the thresholds. Therefore, in addition to the comparison of the suggested method with the optimal solution (which is provided only for small K), we calculate the corresponding lower-bound (22).

We studied more than 200 instances, with different relationships between inventory holding, surplus and shortage costs, update periods and period lengths. Table 2 presents the results for (i) eight-day (long) and four-day (short) update periods; (ii) normal demand probability density with small variance $N(250, 20)$ for long update periods and $N(125, \sqrt{200})$ for short update periods; normal demand probability density with large variance $N(250, 50)$ for long update periods and $N(125, \sqrt{1250})$ for short update periods; (iii) over-capacity with a production rate of 40 product units per day and under-capacity at 30 product units per day.

From Table 2 we observe that the solution obtained with the decomposition approach improves dramatically with the number of updates, K (see Fig. 1, which illustrates typical production conditions). Moreover, even under the worst-case production conditions of short update periods with large variance of demands and over-capacity (which are characterized by a 15.13% gap between the average cost obtained with the

Table 2
Relative gap, %, between the objective function value obtained with the decomposition approach and the lower bound

K	2	3	4	5	6	7	8	9	10
Long period, small variance of demand, over-capacity									
	4.65 (0.54)	3.95	2.00	1.51	1.12	0.76	0.47	0.26	0.03
Long period, small variance of demand, under-capacity									
	1.73 (0.36)	0.94	0.60	0.41	0.29	0.22	0.16	0.12	0.10
Long period, large variance of demand, over-capacity									
	10.11 (2.57)	6.55	4.70	3.62	2.75	1.92	1.24	0.70	0.16
Long period, large variance of demand, under-capacity									
	5.21 (1.20)	3.45	2.60	2.13	1.65	1.34	0.95	0.40	0.19
Short period, small variance of demand, over-capacity									
	7.11 (0.72)	4.59	3.48	2.78	2.33	1.95	1.69	1.47	1.28
Short period, small variance of demand, under-capacity									
	3.63 (0.89)	2.07	1.39	1.00	0.77	0.60	0.49	0.40	0.33
Short period, large variance of demand, over-capacity									
	15.13 (4.56)	10.47 (2.16)	7.98 (1.00)	6.47	5.50	4.66	4.02	3.50	3.04
Short period, large variance of demand, under-capacity									
	8.22 (2.22)	5.48	4.17	3.31	2.79	2.37	2.09	1.86	1.65

The gap between the decomposition approach and the optimal solution is in parentheses.

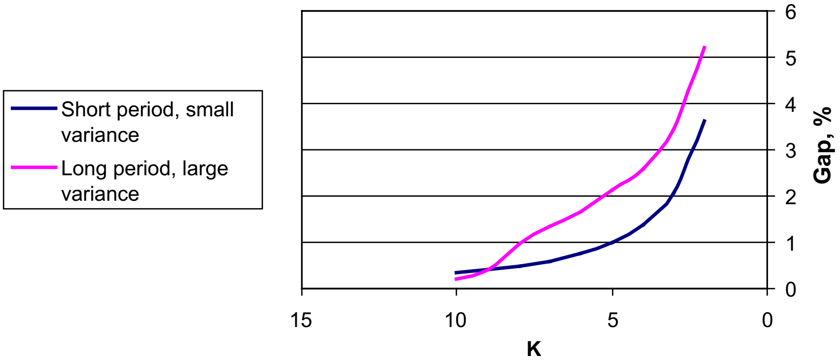

Fig. 1. The gap between the average cost obtained with the decomposition method and the lower bound: under-capacity conditions.

decomposition method and the lower bound), the actual gap, i.e., the gap between the average cost of the optimal solution and that of the decomposition method is less than 5%.

7. Conclusion

This paper addresses a dynamic problem of inventory control under periodic demand update to minimize expected inventory, overproduction and under-production costs with respect to the demand accumulated by the end of the production horizon. Based on a decomposition approach, a feedback policy is derived as either an implicit function of the difference between the current inventory and updated demand level or as an explicit threshold-based control. The policy depends integrally on the number of update periods remaining until the end of the production horizon and implies that only production at a maximum rate (full-load) or no production at all is optimal in the system at each point of time. Moreover, between each consecutive time point of update there can be at most one breaking point at which idling is followed by full-load production.

Our simulation results show that even under the worst-case production conditions of short update periods with large variance of demands and over-capacity, the relative difference between the objective function value of the optimal solution and that of the decomposition approach is less than 5%. Furthermore, the solution obtained with the decomposition approach dramatically improves with the number of updates so that the relative gap reduces to 0.19% when there are more than nine long update periods.

Appendix

Proof of Lemma 2. Consider a solution for system (7) and (8), (10) and (11), which meets the demand, $X(T,\xi)=\sum_{j=1}^{K}D_j$:

$$u(t,\xi)=\begin{cases}0, & \text{for } t_{k-1}\leqslant t<t^*\\ 1, & \text{for } t^*\leqslant t\leqslant T\end{cases},\quad X(T,\xi)=X(t_{k-1},\xi)+U(T-t^*)=\sum_{j=1}^{K}D_j \ \text{ and } \psi(T)=\frac{c}{U}+h(T-t^*).$$

By setting

$$t^*=T-\frac{\sum_{j=1}^{K}D_j-X(t_{k-1},\xi)}{U},$$

we observe that the considered solution is feasible and meets the optimality condition from (13) over the remaining production horizon if $t_{k-1}\leqslant t^*<T$ and $\psi(T)=p$. The former condition evidently holds, if $X(t_{k-1},\xi)<\sum_{j=1}^{K}D_j\leqslant X(t_{k-1},\xi)+U(T-t_{k-1})$ as stated in the lemma. The latter with respect to (11) and $\psi(T)=\frac{c}{U}+h(T-t^*)$ results in an inequality,

$$\psi(T)=\frac{c}{U}+h(T-t^*)\leqslant p^-.$$

Given $p^-\geqslant\frac{c}{U}+hT$, the inequality always holds. □

Proof of Lemma 3. Consider the following under-production solution for system (7) and (8), (10) and (11)

$$\begin{aligned}&u(t,\xi)=U \text{ for } t_{k-1}\leqslant t\leqslant T; X(T,\xi)=X(t_{k-1},\xi)+U(T-t_{k-1});\quad \text{and}\\ &\psi(T)=\psi(t_{k-1})+h(T-t_{k-1})=p^-.\end{aligned}\tag{A1}$$

By setting $\psi(t_{k-1})=-h(T-t_{k-1})+p^-$, we observe that solution (A1) is feasible and meets the full-load condition from (13) over the remaining production horizon if $\psi(t_{k-1})>\frac{c}{U}$, i.e., if $-h(T-t_{k-1})+p^->\frac{c}{U}$ (which always holds) and $\sum_{j=1}^{K}D_j>X(t_{k-1},\xi)+U(T-t_{k-1})$ (under-production), as stated in the lemma. □

Proof of Lemma 4. Given control $u(\tau,\xi)$, $t_{k-1}\leqslant\tau<t_k$, separating over- and under-production in (22), we obtain for $k<K$

$$
\begin{aligned}
J^L = {} & E_{R(k-1,\xi)}\left[\int_0^T [hX(\tau,\xi)+cu(\tau,\xi)]\mathrm{d}t\right] \\
& - \int_{X(t_{k-1},\xi)+U(T-t_k)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau-\sum_{j=1}^{k-1}D_j}^{\infty} p^-\left(X(t_{k-1},\xi)+U(T-t_k)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - w - \sum_{j=1}^{k-1}D_j\right) \\
& \times \phi_k(w)\mathrm{d}w + \int_{X(t_{k-1},\xi)+U(T-t_k)-\sum_{j=1}^{k-1}D_j}^{X(t_{k-1},\xi)+U(T-t_k)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau-\sum_{j=1}^{k-1}D_j} p^+\left(X(t_{k-1},\xi)+U(T-t_k)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - w - \sum_{j=1}^{k-1}D_j\right) \\
& \times \phi_k(w)\mathrm{d}w + \int_{X(t_{k-1},\xi)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau-\sum_{j=1}^{k-1}D_j}^{X(t_{k-1},\xi)+U(T-t_k)-\sum_{j=1}^{k-1}D_j} p^+\left(\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau\right)\phi_k(w)\mathrm{d}w \\
& + \int_{-\infty}^{X(t_{k-1},\xi)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau-\sum_{j=1}^{k-1}D_j} p^+\left(X(t_{k-1},\xi)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - w - \sum_{j=1}^{k-1}D_j\right)\phi_k(w)\mathrm{d}w. \qquad \text{(A2)}
\end{aligned}
$$

Note, that for $k = K$ the last three terms of (A2) merge. Varying the functional (A2) we find

$$
\begin{aligned}
\delta J^L = {} & E_{\xi' \in R(k-1,\xi)}\left[\int_t^T [h\delta X(\tau,\xi') + c\delta u(\tau,\xi')]\mathrm{d}t\right] \\
& - p^-\left(1-\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1}D_j\right)\right)\int_{t_{k-1}}^{t_k} U\delta u(\tau,\xi)\mathrm{d}\tau \\
& + p^+\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1}D_j\right)\int_{t_{k-1}}^{t_k} U\delta u(\tau,\xi)\mathrm{d}\tau \\
& - p^+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau\phi_k\left(X(t_{k-1},\xi)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1}D_j\right)\int_{t_{k-1}}^{t_k} U\delta u(\tau,\xi)\mathrm{d}\tau.
\end{aligned}
$$

Next, taking into account (23)–(25) and requiring $\delta J \geqslant 0$, we obtain the following condition for $k < K$

$$
\begin{aligned}
\delta J^L = {} & \int_t^T h\varepsilon U\delta u\mathrm{d}t + c\varepsilon\delta u - p^-\varepsilon U\delta u \\
& + (p^+ + p^-)\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1}D_j\right)\varepsilon U\delta u \\
& - p^+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau\phi_k\left(X(t_{k-1},\xi)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1}D_j\right)\varepsilon U\delta u \geqslant 0, \qquad \text{(A3)}
\end{aligned}
$$

and for $k = K$,

$$
\delta J^L = \int_t^T h\varepsilon U\delta u\mathrm{d}t + c\varepsilon\delta u - p^-\varepsilon U\delta u + (p^+ + p^-)\Phi_{K-(k-1)}\left(X(t_{k-1},\xi)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1}D_j\right)\varepsilon U\delta u \geqslant 0.
$$

Denote,

$$
\begin{aligned}
\varphi_k(t,\xi) = {} & -h(T-t) + p^- - (p^+ + p^-)\Phi_k\left(X(t_{k-1},\xi)+U(T-t_k)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1}D_j\right) \\
& + p^+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau\phi_k\left(X(t_{k-1},\xi)+\int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1}D_j\right) \quad \text{for } k < K, \qquad \text{(A4)}
\end{aligned}
$$

and

$$\varphi_k(t,\xi) = -h(T-t) + p^- - (p^+ + p^-)\Phi_k\left(X(t_{k-1},\xi) + \int_{t_{k-1}}^{t_k} Uu(\tau,\xi)\mathrm{d}\tau - \sum_{j=1}^{k-1} D_j\right) \quad \text{for } k = K. \tag{A5}$$

Then using (A4) and (A5), we have

$$\delta J^L = \varepsilon U \delta u\left(\frac{c}{U} - \varphi_k(t,\xi)\right) \geqslant 0. \tag{A6}$$

Let the control be maximal at t, $t_{k-1} \leqslant t < t_k$, $u(t,\xi) = 1$. This implies that only negative control variation is possible at this period, $\delta u < 0$. Thus, $u(t,\xi) = 1$ is optimal if condition (A6) holds under $\delta u < 0$, that is, if $\frac{c}{U} - \varphi_k(t,\xi) \leqslant 0$. Similarly $u(t,\xi) = 0$, implies that only $\delta u > 0$ is possible and, thus, from (A6) we find that $u(t,\xi) = 0$, if $\frac{c}{U} - \varphi_k(t,\xi) \geqslant 0$. Finally, both positive and negative variations are feasible when $0 < u(t,\xi) < 1$, which results in $\frac{c}{U} - \varphi_k(t,\xi) = 0$. However by differentiating this condition over an interval of time, we find that $\dot{\varphi}_k(t,\xi) = 0$ over the interval while differentiating Eq. (A4) results in $\dot{\varphi}_k(t,\xi) = h \neq 0$, that is $\frac{c}{U} - \varphi_k(t,\xi) = 0$ cannot hold over an interval of time. □

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