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Optimal co-investment in supply chain infrastructure

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Abstract

This paper considers co-investment in a supply chain infrastructure using an inter-temporal model. We assume that firms' capital is essentially the supply chain's infrastructure. As a result, firms' policies consist in selecting an optimal level of employment as well as the level of co-investment in the supply chain infrastructure. Several applications and examples are presented and open-loop, as well as feedback solutions are found for non-cooperating firms, long- and short-run investment cooperation and non-simultaneous moves (Stackelberg) firms. In particular, we show that a solution based on Nash and Stackelberg differential games provides the same level of capital investment. Thus, selecting the leader and the follower in a co-investment program does not matter. We show that in general, co-investments by firms vary both over time and across firms, and thereby render difficult the implementation of co-investment programs for future capital development. To overcome this problem, we derive conditions for firms' investment share to remain unchanged over time and thus be easily planned.

Keywords: Supply chain management; Gaming; Investment analysis; Control

1. Introduction

Logistics is the management of means applied to some end. It is subjugated to needs that underlie the flow of goods, transports, information and operations in general. For example, an investment in ports is an investment in logistic infrastructure that benefits its stakeholders, "a supply chain" or a network of firms and consumers who use the port. These investments are needed to manage the physical flow of goods across navigation lines, allowing thereby an exchange between firms who require the port and its associated infrastructures. Similarly, investments in computer communication systems, telephony, inventory systems, etc. are enablers, providing the means for traffic flow or providing an ability to meet demands when they occur. In a similar sense, a Supply Chain Infrastructure provides the means for the chain economic entities or firms sharing a common interest to participate in a mutual exchange. The efficient delivery of such exchanges requires co-investments in infrastructure however. Examples to these effects abound and assume many forms. For example, collective farms in Israel have long co-invested and used common logistic and capital equipment (tractors, common marketing facilities, etc.) to exploit their land and the fruits of their labor. By the same token, in elementary supplier–producer relationships, investments in Electronic Data Interchange (EDI) have provided for both firms the ability to improve the quality of their data exchange through computers and thereby their profitability. These problems as well as many issues related to supply chain management have attracted recently a great deal of attention. For example, Gunasekaran (2004), Meskens and Riane (2006) and Stadtler (2005) are editorial on special issues as well as reviews (albeit none has addressed issues of infrastructure investment in supply chain).

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These investments in infrastructure, although costly and often difficult to justify, are assumed essential, without which a collective farm, a logistic infrastructure or a supply chain is not sustainable. Infrastructure co-investments are difficult to assess and value however. In the Transport industry, numerous contributions have been made to assess specific projects (Adler, 1987; Button and Pearman, 1983; Gillen, 1996; Nijkamp and Blaas, 1994 are some examples). The interest in the productivity of infrastructure macroeconomic investments has attracted immense attention in the early 1990s following seminal econometric research by Aschauer (1989a,b) (see also Munnell, 1992) who pointed out to an important productivity benefit and thereby justifying immense investment in infrastructures. Related studies include Berndt and Hansson (1991) on infrastructure investment in Sweden, a survey by Gramlich (1994) on infrastructure investment, as well as Holtz and Schwartz (1995) and Sanchez-Robles (1998) on infrastructure and economic growth. The carry over to investments in supply chain infrastructures by interacting firms, sharing wholly or partly common goals and information has not received any particular attention however. This is despite the fact that, there is an extensive literature based on game theory which recognizes the intricate and the interdependent relationships that exist between members of supply chains (for example, see Cachon and Netessine, 2004; Kogan and Tapiero, 2007).

Game theory models are broadly employed to study interactions between firms pertaining to contract negotiations, collaborative strategies, outsourcing, cost reduction and capacity investing (Kamien and Li, 1990; Venkatesan, 1992; Mieghem, 1999). For example, Mieghem (1999) builds a game theoretic model on production outsourcing with respect to investment decisions. The goal is to reduce all decentralization costs and coordinate capacity *investment*. The use of assets with regard to ownership is the focus of Grossman and Hart (1986). These investment and asset related research directions, however, do not account for their impact on commonly shared infrastructure by firms in supply chains.

The ability to collect detailed information about customer behavior and the ease of communication due to new technologies (including Internet and IT) have motivated extensive research into dynamic decision making in general and continuous-time production, pricing and investment strategies in particular (see, for example, recent surveys by Elmaghraby and Keskinocak (2003) and by Chan et al. (2003)). However, due to mathematical difficulties inherent in differential games, i.e., games involving decisions made continuously, the supply chain management literature has been primarily concerned only with applying deterministic differential games between two interacting firms (Cachon and Netessine, 2004).

In this paper we both formalize the concern for investments and at the same time provide some theoretical insights based on an inter-temporal differential game of co-investment by multiple firms in infrastructure (for broadly related approaches in supply chain management see also Li and Wang, 2007; Nagarajan and Sosic, 2006; Lau et al., 2007, as well as Hammond and Beullens, 2007).

Our analysis proceeds as follows. First we consider an N-person differential investment game model and calculate the open- and closed-loop Nash strategies for each of the firms in the supply chain and assess their implications. Subsequently, we consider different forms of cooperation including a centralized supply chain as well as power asymmetry between the firms modeled by Stackelberg games and compare their implications for co-investment in the supply chain infrastructure. Such an approach makes it possible to analyze and compare collaboration under different organizational structures. Examples are solved and used to demonstrate and validate our results. Most of our proofs are relegated to a Appendix for exposition purposes however. Further discussion of the implications of our results, are developed in our conclusions.

2. A dynamic model for co-investment in infrastructure

Consider N-firms operating in a supply chain, each characterized by its output price $p_j(t)$ at time t, labor force $L_j(t)$, investment policy $I_j(t)$ and an aggregate production function $Q = f(K, L_j)$, $\frac{\partial f}{\partial K} \geqslant 0$, $\frac{\partial f}{\partial L_j} > 0$ for $L \neq 0$, $\frac{\partial f(K,0)}{\partial L_j} = 0$, $\frac{\partial^2 f}{\partial L_j^2} \leqslant 0$, $\frac{\partial^2 f}{\partial K^2} \leqslant 0$. We let $K(t)$ be the level of current supply chain infrastructure capital, deteriorating at the rate δ. The process of capital accumulation is then given by

$$\frac{\mathrm{d}K(t)}{\mathrm{d}t} = -\delta K(t) + \sum_{j=1}^{N} I_j(t), \quad K(0) = K_0,$$
$$I_j(t) \geqslant 0, \ j = 1, \ldots, N. \tag{1}$$

The firms' objective consists in maximizing the discounted profit by selecting an optimal employment policy on the one hand and a co-investment in supply chain infrastructure (contributing thereby to all firms potential revenues) on the other. The objective is specified by

$$\operatorname*{Max}_{L_j, I_j} \int_0^\infty \mathrm{e}^{-r_j t}[p_j(t) f(K(t), L_j(t)) - c_j(t) L_j(t) - C_{Ij}((1-\theta) I_j(t))]\mathrm{d}t, \quad j = 1, \ldots, N, \tag{2}$$

where r_j is the discount rate, $c_j(t)$ is the labor cost and $C_{Ij}(\cdot)$ is a continuous, twice differentiable and increasing investment cost function, $\frac{\partial C_{Ij}}{\partial I_j} > 0$, $\frac{\partial^2 C_{Ij}}{\partial I_j^2} \geqslant 0$, mitigated by a proportion which is subsidized and given by θ. To study the problem, we construct the Hamiltonians (see Tapiero, 1988, 1998):

$$H_j(t) = \mathrm{e}^{-r_j t}[p_j(t) f(K(t), L_j(t)) - c_j(t) L_j(t) - C_{Ij}((1-\theta) I_j(t))] + \psi_j(t)\left(\sum_{j=1}^{N} I_j(t) - \delta K(t)\right), \tag{3}$$

where the co-state variables are determined by

$$\dot{\psi}_j(t) = -e^{-r_j t} p_j(t) \frac{\partial f(K(t), L_j(t))}{\partial K(t)} + \delta \psi_j(t),$$
$$\lim_{t\to\infty} \psi_j(t) = 0. \tag{4}$$

The Hamiltonian (3) can be interpreted as the instantaneous profit rate of firm j, which includes the firm j value $\psi_j(t)\dot{K}(t)$ of increment $\dot{K}(t)$ in the infrastructure capital. The co-state variable $\psi_j(t)$ is the shadow price, i.e., the net benefit of firm j from investing one more monetary unit at time t. The differential equation (4) states that the marginal opportunity cost $\delta\psi_j(t)$ of investment of firm j in infrastructure should equal the (discounted) marginal profit $e^{-r_j t} p_j(t) \frac{\partial f(K(t), L_j(t))}{\partial K(t)}$ from increased productivity and from the capital gain $\dot{\psi}_j(t)$.

Optimal policies are found by maximizing the Hamiltonians with respect to investments, $I_j(t)$, and labor, $L_j(t)$, which yields:

$$L_j(t) = \begin{cases} l_j(t), & \text{if } \frac{\partial f(K(t), L_j(0))}{\partial L_j} \leqslant \frac{c_j(t)}{p_j(t)}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $l_j(t)$ is determined by

$$\frac{\partial f(K(t), l_j(t))}{\partial L_j} - \frac{c_j(t)}{p_j(t)} = 0, \tag{6}$$

and

$$I_j(t) = \begin{cases} i_j(t), & \text{if } \psi_j(t) \geqslant \frac{\partial C_{Ij}(0)}{\partial I_j} e^{-r_j t}, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

with $i_j(t)$ determined by

$$\psi_j(t) = \frac{\partial C_{Ij}((1-\theta) i_j(t))}{\partial I_j} e^{-r_j t}. \tag{8}$$

Since the objective function (2) is concave and constraints (1) are linear, conditions (5)–(8) are necessary and sufficient for optimality and will be considered next in detail, providing specific insights regarding the investment process in supply chain infrastructure. We consider first the firm Nash strategies.

3. The N-firms open-loop Nash strategies

The Nash equilibrium for each firm is obtained by optimizing simultaneously all N Hamiltonians (4). These result in the following lemma.

Lemma 1. *If $a(t)$, $b_j(t)$ and $i_j(t)$, $j=1,\ldots,N$ satisfy the following system of equations*:

$$\frac{da(t)}{dt} = -\delta a(t) + \sum_{j=1}^{N} i_j(t), \quad a(0) = K_0,$$
$$b_j(t) = \frac{\partial C_{Ij}((1-\theta) i_j(t))}{\partial I_j} e^{-r_j t},$$
$$\frac{\partial f(a(t), l_j(t))}{\partial L_j} = \frac{c_j(t)}{p_j(t)},$$
$$\dot{b}_j(t) = -e^{-r_j t} p_j(t) \frac{\partial f(a(t), l_j(t))}{\partial a(t)} + \delta b_j(t), \quad \lim_{t\to\infty} b_j(t) = 0,$$
$$j = 1, \ldots, N,$$

then, the pair of dynamic strategy sets $\{I_j^n(t) = i_j(t), j=1,\ldots,N\}$ and $\{L_j^n(t) = l_j(t), j=1,\ldots,N\}$, along with the capital $K^n = a(t)$, $t \geqslant 0$ is a Nash equilibrium in the supply chain co-investment and labor force differential game.

Proof. See Appendix. □

The implications of this lemma are best examined through an example which assumes a Cobb–Douglas Production function and a quadratic investment cost.

Example 1. Let the aggregate production function be a Cobb–Douglas function, $f(K, L_j) = aK^{\alpha} L_j^{\beta}$, with $\alpha + \beta = 1$, $C_{Ij}(I) = c_{Ij}(1-\theta) I_j^2$ and let the labor cost increase slower than the price index raised to power β so that:

$$\omega_j(t) = \left[\frac{p_j(t)}{c_j^{\beta}(t)}\right]^{\frac{1}{1-\beta}} = e^{\varepsilon t}, \quad \varepsilon < \min\{r_j, j = 1, \ldots, N\}.$$

Using (6), we have $aK^{\alpha}(t)\beta l_j^{\beta-1}(t) - \frac{c_j(t)}{p_j(t)} = 0$, and thus,

$$L_j(t) = l_j(t) = \left[\frac{1}{a\beta} \frac{c_j(t)}{p_j(t) K^{\alpha}(t)}\right]^{\frac{1}{\beta-1}}.$$

Note, that $\frac{\partial L_j(t)}{\partial K(t)} > 0$, if $\beta < 1$ and $\frac{\partial L_j(t)}{\partial K(t)} < 0$, if $\beta > 1$.

Next, from Lemma 1, we have

$$\dot{b}_j(t) = -e^{-r_j t} p_j(t) a\alpha a^{\alpha-1}(t) l_j^{\beta}(t) + \delta b_j(t)$$
$$= -e^{-r_j t} \left[\frac{p_j(t)}{c_j^{\beta}(t)}\right]^{\frac{1}{1-\beta}} \frac{\alpha}{\beta} [a\beta]^{\frac{1}{1-\beta}} a^{\frac{\alpha}{1-\beta}-1}(t) + \delta b_j(t),$$

which with respect to $\alpha + \beta = 1$ and $\omega_j(t) = \left[\frac{p_j(t)}{c_j^{\beta}(t)}\right]^{\frac{1}{1-\beta}} = e^{\varepsilon t}$, results in

$$\dot{b}_j(t) = -e^{-(r_j - \varepsilon)t} \frac{\alpha}{\beta} [a\beta]^{\frac{1}{1-\beta}} + \delta b_j(t).$$

Noting that $\lim_{t\to\infty} b_j(t) = 0$, we find:

$$b_j(t) = \frac{e^{-(r_j-\varepsilon)t}}{r_j - \varepsilon + \delta} \frac{\alpha}{\beta} [a\beta]^{\frac{1}{1-\beta}}.$$

Solving $b_j(t) = \frac{\partial C_I((1-\theta) i_j(t))}{\partial I_j} e^{-r_j t}$ in $i_j(t)$ we find optimal investment strategies for each of the firms, j:

$$I_j^n(t) = i_j(t) = \frac{e^{\varepsilon t}}{r_j - \varepsilon + \delta} \frac{\alpha}{2(1-\theta) c_{Ij} \beta} [a\beta]^{\frac{1}{1-\beta}}, \quad j = 1, \ldots, N.$$

The total supply chain capital is then obtained from Lemma 1, by

$$\frac{da(t)}{dt} = -\delta a(t) + \sum_{j=1}^{N} i_j(t), \quad \text{or}$$
$$\dot{a}(t) = -\delta a(t) + \sum_{j=1}^{N} \frac{e^{\varepsilon t}}{r_j - \varepsilon + \delta} \frac{\alpha}{2(1-\theta) c_{Ij} \beta} [a\beta]^{\frac{1}{1-\beta}}.$$

The solution of this differential equation yields the supply chain capital explicitly given by

$$K^n(t) = a(t) = \frac{1}{(\varepsilon+\delta)}\frac{\alpha}{2(1-\theta)\beta}[a\beta]^{\frac{1}{1-\beta}}\sum_{j=1}^{N}\frac{1}{2(r_j-\varepsilon+\delta)c_{Ij}}\mathrm{e}^{\varepsilon t} + A\,\mathrm{e}^{-\delta_t},$$

where A is determined by the boundary condition $a(0)=K_0$,

$$A = K_0 - \frac{1}{(\varepsilon+\delta)}\frac{\alpha}{2(1-\theta)\beta}[a\beta]^{\frac{1}{1-\beta}}\sum_{j=1}^{N}\frac{1}{2(r_j-\varepsilon+\delta)c_{Ij}}.$$

This solution implies that the growth of equilibrium investments over time is inversely proportional to the firms' discount rates and investment costs. This strategy compensates the effect of price index increases over weighted labor costs as shown in Fig. 1. Further, we have $\frac{\partial I_j}{\partial\theta}>0$, $\frac{\partial^2 I_j}{\partial\theta^2}>0$, meaning that the larger the subsidies the larger the co-investments, growing then at an increased rate. For this reason, supply chain support for individual member firm co-investment is indeed important and may justify in some cases a "centralized control" which dictates to member firms the intensity of their investment. The level of capital will thus increase as well as a function of the support parameter.

Fig. 1 above points out to the optimal equilibrium over time when $\omega(t)=\mathrm{e}^{\varepsilon t}$, $r_1>r_2$, $c_{I1}>c_{I2}$, and $\left[\frac{p_1(t)}{c_1^{\beta}(t)}\right] < \left[\frac{p_2(t)}{c_2^{\beta}(t)}\right]$. Thus, if firms comprising the supply chain differ in their basic parameters and at least some of the parameters (price index, labor cost, investment costs and so on) change in time, then firms' investments not only differ and change over time, but their co-investment shares in the overall infrastructure capital may diverge in time as well. Such a change in investment strategies thus makes it difficult to plan future capital development of the supply chain. For this reason, a strategy that imposes steady co-investment shares by firms might lead to results that are viable. We shall turn our attention next to this special case.

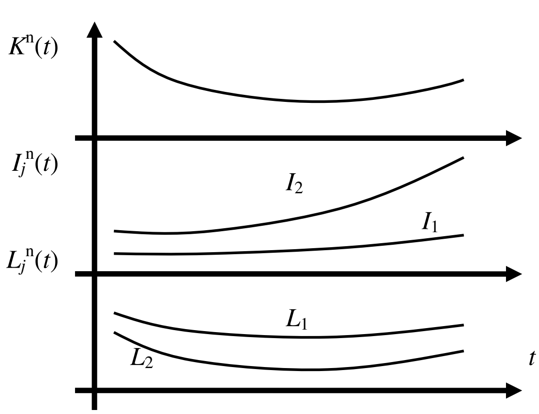

Fig. 1. The Nash equilibrium over time, the case of $N=2$.

Lemma 2. *Let* $p_j(t)\frac{\partial f(\overline{K},l_j(t))}{\partial K}\Big/\frac{\partial C_{Ij}((1-\theta)\overline{i}_j)}{\partial I_j}=\vartheta_j$ *and* $\overline{K}$, $\overline{i}_j$, $l_j(t)$, $j=1,\dots,N$, *satisfy the following equations for* $t\geqslant 0$:

$$p_j(t)\frac{\partial f(\overline{K},l_j(t))}{\partial K} - \frac{\partial C_{Ij}((1-\theta)\overline{i}_j)}{\partial I_j}(r_j+\delta)=0,$$

$$\sum_{j=1}^{N}\overline{i}_j=\delta\overline{K},\quad \frac{\partial f(\overline{K},l_j(t))}{\partial L_j}=\frac{c_j(t)}{p_j(t)},\quad j=1,\dots,N.$$

If $K_0=\overline{K}$, *then there exists the pair of strategy sets: static investment* $\{Ij_n(t)=\overline{i}_j,\ j=1,\dots,N\}$ *and employment* $\{L_j^n(t)=l_j(t), j=1,\dots,N\}$, $t\geqslant 0$ *which is a Nash equilibrium in the supply chain investment and labor force differential game.*

Proof. See Appendix. □

Below, we consider an example to highlight the effects of a constant co-investment strategy.

Example 2. Let the aggregate production function be again the Cobb–Douglas function, $f(K,L_j)=aK^{\alpha}L_j^{\beta}$, with $\alpha+\beta=1$, $C_{Ij}(I)=c_{Ij}(1-\theta)I_j^2$, $r_j=r$ for $j=1,\dots,N$, $K_0=\overline{K}$, and $\omega_j=\left[\frac{p_j(t)}{c_j^{\beta}(t)}\right]^{\frac{1}{1-\beta}}$. Using $\frac{\partial f(\overline{K},l_j(t))}{\partial L_j}=\frac{c_j(t)}{p_j(t)}$ of Lemma 2, we have:

$$L_j(t)=\overline{l}_j(t)=\left[\frac{1}{a\beta}\frac{c_j(t)}{p_j(t)\overline{K}^{\alpha}}\right]^{\frac{1}{\beta-1}}=\overline{K}\left[\frac{1}{a\beta}\frac{c_j(t)}{p_j(t)}\right]^{\frac{1}{\beta-1}}.$$

Taking into account,

$$p_j(t)\frac{\partial f(\overline{K},l_j(t))}{\partial K}+\frac{\partial C_{Ij}((1-\theta)\overline{i}_j)}{\partial I_j}(r+\delta)=0\quad\text{and}$$

$$\sum_{j=1}^{N}\overline{i}_j=\delta\overline{K},$$

of Lemma 2, we obtain a system of $N+1$ equations in $N+1$ unknowns, $\overline{i}_j$, $j=1,\dots,N$, $\overline{K}$:

$$w_j\frac{\alpha}{\beta}[a\beta]^{\frac{1}{1-\beta}}\overline{K}^{\frac{\alpha}{1-\beta}-1}-2c_{Ij}(1-\theta)(r+\delta)\overline{i}_j=0,\qquad \sum_{j=1}^{N}\overline{i}_j=\delta\overline{K}.$$

Summing all equations, we have:

$$\sum_{j=1}^{N}\left[\frac{w_j}{c_{Ij}}\right]\frac{\alpha}{\beta}[a\beta]^{\frac{1}{1-\beta}}\overline{K}^{\frac{\alpha}{1-\beta}-1}-2(1-\theta)(r+\delta)\delta\overline{K}=0,$$

which together with $\alpha+\beta=1$ results a constant co-investment strategy when $K_0=\overline{K}$:

$$\overline{K}=\frac{\sum_{j=1}^{N}\left[\frac{w_j}{c_{Ij}}\right]\frac{\alpha}{\beta}[a\beta]^{\frac{1}{1-\beta}}}{2(1-\theta)(r+\delta)\delta}\quad\text{and}\quad \overline{i}_j=\frac{w_j\frac{\alpha}{\beta}[a\beta]^{\frac{1}{1-\beta}}}{2c_{Ij}(1-\theta)(r+\delta)},$$

$j=1,\dots,N.$

Interestingly, note that $\frac{\partial\overline{K}}{\partial\theta}>0$ and $\frac{\partial\overline{i}}{\partial\theta}>0$ which points out to a growth of capital and co-investment when investment subsidies increase.

What if a sustainable $\overline{K}$ exists and is attainable, but $K_0 \neq \overline{K}$? There are two possible approaches to dealing with this problem.

The first approach is non-cooperative and readily emanates from Lemmas 1 and 2. Indeed, one can assume that there is a point in time, say t^*, at which the supply chain attains a stationary investment equilibrium (described in Lemma 2) as t^* tends to infinity. To determine whether such an equilibrium exists, the system of equations stated in Lemma 1 can be resolved for the terminal condition, $K(t^*) = \overline{K}$. The solution that Lemma 1 thus provides is a dynamic, open-loop Nash equilibrium. While such a solution allows to gain some insights, it is difficult to implement. If the firms do not collaborate, then a closed-loop solution may be more viable. We develop such a solution in the next section.

The other approach consists in using open-loop policies for cooperating, which can be short- and long-run as discussed in Sections 5 and 6.

4. The *N*-firms feedback Nash strategy

In this section, we show how to obtain a closed-loop equilibrium in the conditions of Lemma 2, i.e., when a stationary investment equilibrium is attainable. The derivation is accomplished by employing an equivalent formulation of the maximum principle. Specifically, let $\Psi_j(t) = \psi_j(t)\mathrm{e}^{r_jt}$. Then $\psi_j(t) = \Psi_j(t)\mathrm{e}^{-r_jt}$ and $\dot{\psi}_j(t) = \mathrm{e}^{-r_jt}(\dot{\Psi}_j(t) - r_j\Psi_j(t))$. Using these notations in conditions of Lemma 2, the co-state Eq. (4) and the optimality condition (8) take the following form, respectively,

$$\dot{\Psi}_j(t) - r_j\Psi_j(t) = -p_j(t)\frac{\partial f(K(t), L_j(t))}{\partial K(t)} + \delta\Psi_j(t),$$
$$\lim_{t\to\infty} \Psi(t)\mathrm{e}^{-r_jt} = 0, \tag{9}$$
$$\Psi_j(t) = \frac{\partial C_{Ij}((1-\theta)i_j(t))}{\partial I_j}. \tag{10}$$

Denote the solution of Eq. (10) as $i_j(t) = F_j(\Psi_j(t))$. To simplify the presentation we next suppress index t wherever the dependence on time is obvious. Consequently, the stationary investment conditions are $\dot{K} = 0$ and $\dot{\Psi} = 0$, and from (9) the static co-state value $\overline{\Psi}_j$ of the co-state variable is fined by

$$\overline{\Psi}_j = \frac{p_j}{\delta + r_j}\frac{\partial f(\overline{K}, \bar{l}_j)}{\partial K}, \tag{11}$$

as well as the steady-state capital is equal to $\overline{K}$ (see Lemma 2). Let us introduce a new function, $\Phi_j(\cdot)$,

$$\Psi_j(t) = \Phi_j(K(t)). \tag{12}$$

Denote the solution of Eq. (6) as $l_j = F_{Lj}(K)$. Differentiating (12) we have $\dot{\Psi}_j = \Phi'_j(K)\dot{K}$, which when substituting the state (1) and co-state (9) equations leads to,

$$-p_j\frac{\partial f(K, F_{Lj}(K))}{\partial K} + (\delta + r_j)\Phi_j(K)$$
$$= \Phi'_j(K)\left[\sum_{j=1}^{N} F_j(\Phi_j(K)) - \delta K\right], \Phi_j(\overline{K}) = \overline{\Psi}_j.$$

Thus, we have proved the following theorem.

Theorem 1. *If $p_j\frac{\partial f}{\partial K}/\frac{\partial C_{Ij}}{\partial I_j}$ does not explicitly depend on time for $j=1,\ldots,N$, then investment $\{I_j^n = F_j(\Phi_j(K)), j=1,\ldots,N\}$ and employment $\{L_j^n(t) = F_{Lj}(K), j=1,\ldots,N\}$, $t \geqslant 0$ constitute a feedback Nash equilibrium in the supply chain investment and labor force differential game, where $\Phi_j(K)$, $j = 1,\ldots,N$ satisfy the following differential equations,*

$$\Phi'_j(K)\left[\sum_{j=1}^{N} F_j(\Phi_j(K)) - \delta K\right] + p_j\frac{\partial f(K, F_{Lj}(K))}{\partial K}$$
$$-(\delta + r_j)\Phi_j(K) = 0, \quad \Phi_j(\overline{K}) = \overline{\Psi}_j,\ j = 1,\ldots,N. \tag{13}$$

The following example illustrates the results of Theorem 1.

Example 3. Assume the conditions of Example 2, except $K_0 < \overline{K}$ and $\alpha + \beta < 1$. Then we have:

$$I_j^n = \frac{\Psi_j}{2c_{Ij}(1-\theta)} = \frac{\Phi_j(K)}{2c_{Ij}(1-\theta)},$$
$$\overline{K} = \left[\frac{\sum_{j=1}^{N}\left[\frac{p_j}{c_{Ij}}\right]\left[a\beta\frac{p_j}{c_j}\right]^{\frac{\beta}{1-\beta}}}{2\delta(1-\theta)(r+\delta)}\right]^{\frac{1-\beta}{2-2\beta-\alpha}} \quad \text{and} \quad L_j^n = K^{\frac{\alpha}{1-\beta}}\left[\frac{1}{a\beta}\frac{c_j}{p_j}\right]^{\frac{1}{\beta-1}}.$$

As a result, the system of backward differential equations (13) takes the following form:

$$\Phi'_j(K)\left[\sum_{j=1}^{N}\frac{\Phi_j(K)}{2(1-\theta)c_{Ij}} - \delta K\right] + \xi_jK^{\frac{\alpha}{1-\beta}-1}$$
$$-(\delta + r)\Phi_j(K) = 0,$$
$$\Phi_j(\overline{K}) = \frac{\xi_j}{\delta + r}\overline{K}^{\frac{\alpha}{1-\beta}-1},\ j = 1,\ldots,N, \tag{14}$$

where $\xi_j = p_ja\alpha\left[\frac{1}{a\beta}\frac{c_j}{p_j}\right]^{\frac{\beta}{\beta-1}}$.

We solve this system of equations with Maple for two firms, $N = 2$, $a = 1$, $\alpha = 0.1$, $\beta = 0.1$, $\theta = 0.4$, $\delta = 0.04$, $r_1 = r_2 = 0.002$, $c_1 = 0.4$, $c_2 = 0.5$, $c_{I1} = 0.2$, $c_{I2} = 0.3$, $p_1 = p_2 = 7$.

The resultant feedback policies of the two firms, $I_1^n = \frac{\Phi_1(K)}{2c_{I1}(1-\theta)}$ and $I_2^n = \frac{\Phi_2(K)}{2c_{I2}(1-\theta)}$, is illustrated graphically in Fig. 2. The corresponding evolution in time of the capital, $K^n(t)$, $\frac{\mathrm{d}K^n(t)}{\mathrm{d}t} = -\delta K^n(t) + \sum_{j=1}^{N} I_j^n(t)$ and investments for the case of $K(0) = 0.2 < \overline{K} = 69.91217939$ are depicted in Figs. 3 and 4, respectively.

From Figs. 2–4, we observe that the greater the capital, the lower the investments. When the infrastructure capital is greater (smaller) than the steady-state level $\overline{K}$, it is optimal to invest in total by all firms less (more) than $\delta\overline{K}2.796487176$, so that the overall accumulated capital decreases (increases) towards the stationary investment

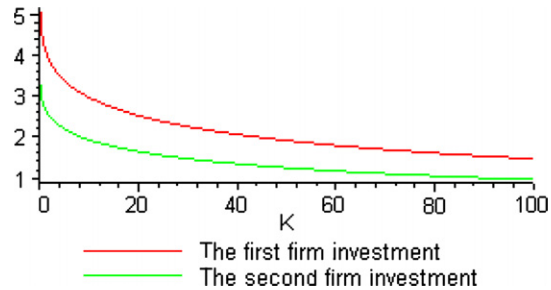

Fig. 2. The feedback equilibrium investments as a function of capital, $I_1^n = F_1(\Phi_1(K))$ and $I_2^n = F_2(\Phi_2(K))$.

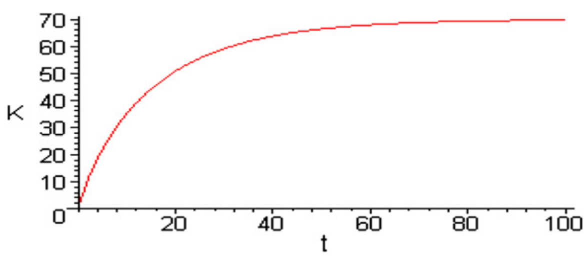

Fig. 3. Evolution of the capital over time, $K^n(t)$.

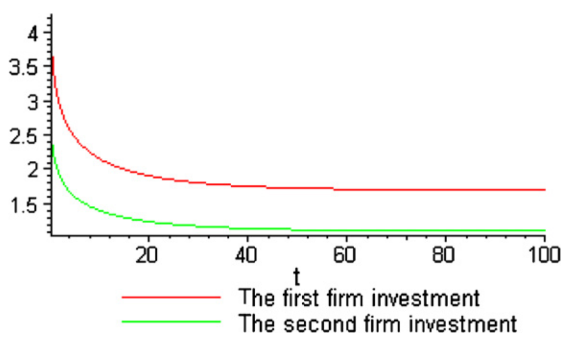

Fig. 4. Evolution of the investment over time, $I_1^n(t)$ and $I_2^n(t)$.

equilibrium. Furthermore, the investments decrease much faster when the capital exceeds the static level compared to the rate of their decrease when the capital is lower than the static level (see Fig. 2).

Naturally, the first firm, which has lower investment cost, $c_{I1} = 0.2 < c_{I2} = 0.3$ invests more than the second firm (see Figs. 2).

Since $\frac{c_j}{p_j}$ is constant in the example, the static capital, $\overline{K}$, induces the equilibrium employment to attain a static level as well, $\overline{L}_j = \overline{K}^{\frac{\alpha}{1-\beta}}\left[\frac{1}{a\beta}\frac{c_j}{p_j}\right]^{\frac{1}{\beta-1}}$. The evolution in time of the equilibrium employment for the two firms is shown in Fig. 5.

From Fig. 5, we observe that the employment increases with the capital and it tends to the static level for the firms, $\overline{L}_1 = 2.98533$ and $\overline{L}_2 = 2.329779$, which is higher for the first firm as its wages are lower, $c_1 = 0.4 < c_2 = 0.5$. Since employment is proportional to the infrastructure capital, the rate of employment changes much faster when the infrastructure capital is low.

Fig. 5. Evolution of the employment over time, $L_1^n(t)$ and $L_2^n(t)$.

5. Short-run cooperation

If all parties are interested in a stationary co-investment strategy for the supply chain (and therefore are seeking a stationary equilibrium, see Lemma 2), it can be implemented by determining jointly time t^*, and collaborative investment policies, $I_j(t)$ for $0 \leqslant t \leqslant t^*$, $j = 1,\ldots,N$. This is accomplished by requiring that the joint-capital, $K(t)$, will reach the desired optimal level $\overline{K}$ by t^*, i.e., $K_0 + \sum_{j=1}^{N}\int_0^{t^*} I_j(t)\mathrm{d}t = \overline{K}$ (as shown in Fig. 6).

Furthermore, inserting ω_j in the infrastructure capital we obtain an optimal balance between the infrastructure capital and total weighted firms' prices as shown below:

$$\overline{K} = \frac{\alpha[a\beta]^{\frac{1}{1-\beta}}\sum_{j=1}^{N}\left[\frac{p_j(t)}{c_j^{\beta}(t)}\right]\frac{1}{c_{Ij}}}{2\beta(1-\theta)(r+\delta)\delta} = \rho\sum_{j=1}^{N}\left[\frac{p_j(t)}{c_j^{\beta}(t)}\frac{1}{c_{Ij}}\right],$$

where ρ is a constant and a firm's price is proportional to the labor cost, $p_j(t) = w_j c_j^{\beta}(t)$. Consequently, if all firms have the same labor cost, $c_j(t) = c(t)$, j=1,…,N, then the supply chain price is $p(t) = \frac{\overline{K}}{\bar{\rho}}c^{\beta}(t)$, $\bar{\rho} = \rho\sum_{j=1}^{N}\frac{1}{c_{Ij}}$ and the optimal level of fixed capital is inversely proportional to the labor cost.

An ultimate way of short-run cooperation is a one-time partnership. Assume that firms agree to cooperate until a common point t^* to reach the stationary equilibrium in minimum time. Optimal control theory shows that in such a case one time, high level investments, $\widehat{I}_j$, j=1,…,N will be optimal, so that $\sum_{j=1}^{N}\widehat{I}_j = \overline{K} - K_0$.

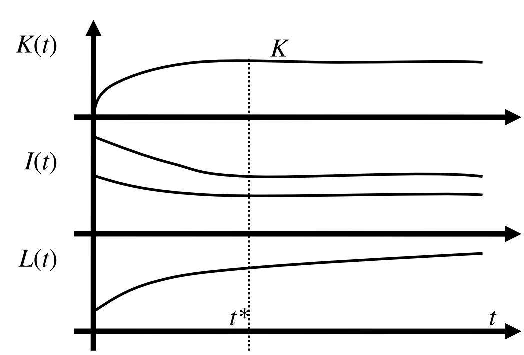

Fig. 6. The collaborative solution over time, the case of $K_0 < \overline{K}$ and $N = 2$.

To model an instantaneous investment, $\widehat{I}_j$, we employ the Dirac delta function $\Delta(t)$, $I_j(t) = \widehat{I}_j\Delta(t)$. Specifically, the optimal policy for reaching the desired equilibrium in minimum time must be a solution of the following optimization problem,

$$\min_{I_j} t^*$$
$$\text{s.t.}\quad \frac{\mathrm{d}K(t)}{\mathrm{d}t} = -\delta K(t) + \sum_{j=1}^{N} I_j(t), \qquad (15)$$
$$K(0) = K_0, \quad K(t^*) = \overline{K}, \quad I_j(t) \geqslant 0,$$
$$j = 1,\ldots,N, \quad 0 \leqslant t \leqslant t^*.$$

Using the maximum principle we construct the Hamiltonian

$$H(t) = \lambda(t)\left(-\delta K(t) + \sum_{j=1}^{N} I_j(t)\right), \qquad (16)$$

where the co-state variable is determined by

$$\dot{\lambda}(t) = \delta\lambda(t). \qquad (17)$$

Furthermore, since t^* is unknown, an additional necessary transversality condition is that, $H(t) = 1$, i.e.,

$$\lambda(t)\left(-\delta K(t) + \sum_{j=1}^{N} I_j(t)\right) = 1. \qquad (18)$$

Maximizing the Hamiltonian (16), we readily observe that if $\lambda(t) < 0$, then no investment is optimal, if $\lambda(t) = 0$, then the investments are arbitrary non-negative values. Otherwise, if $\lambda(t) > 0$, then $I_j(t) \to +\infty$, $j = 1,\ldots,N$. Based on these properties, we have the following result.

Lemma 3. *The optimal solution of problem (15) is $t^* = 0$ and $I_j(t^*) = \widehat{I}_j\Delta(t^*)$, $j = 1,\ldots,N$, where $\Delta(t^*)$ is Dirac delta function at t^* and $\sum_{j=1}^{N}\widehat{I}_j = \overline{K} - K_0$.*

Proof. See Appendix. □

Note that a stationary equilibrium can be viewed as both open- and closed-loop equilibrium. From Lemma 3 it follows that, if *competing* firms are able to *cooperate* in setting their one-time investments, then the firms can reach the stationary Nash equilibrium in *no-time* and stay there infinitely long as summarized in the following theorem.

Theorem 2. *Assume that at some point of time $t = t^*$ there exists a sustainable level of the capital $K(t^*)$. Let $p_j(t)\frac{\partial f(\overline{K}, l_j(t))}{\partial K}\Big/\frac{\partial C_{Ij}((1-\theta)\bar{i}_j)}{\partial I_j} = \vartheta_j$ and $\overline{K}$, $\bar{i}_j$, $l_j(t)$, $j = 1,\ldots,N$, satisfy the following equations for $t \geqslant t^*$:*

$$p_j(t)\frac{\partial f(\overline{K}, l_j(t))}{\partial K} - \frac{\partial C_{Ij}((1-\theta)\bar{i}_j)}{\partial I_j}(r_j + \delta) = 0,$$
$$\sum_{j=1}^{N}\bar{i}_j = \delta\overline{K}, \quad \frac{\partial f(\overline{K}, l_j(t))}{\partial L_j} = \frac{c_j(t)}{p_j(t)}, \ j = 1,\ldots,N.$$

If $K(t^*) = \overline{K}$, then there exists the pair of strategy sets: static investment $\{Ij_n(t) = \bar{i}_j, j=1,\ldots,N\}$ and employment $\{Lj^n(t) = l_j(t),\ j=1,\ldots,N\}$, $t \geqslant 0$ which is a Nash equilibrium in the supply chain investment and labor force differential game for $t \geqslant t^*$.

Proof. The proof is similar to Lemma 2 and therefore omitted. □

6. Long-run cooperation and the centralized solution

In this section, we shall consider a centralized, supply chain co-investment strategy, which turns out, expectedly, to be different than the Nash strategy obtained earlier. In this organizational mode the supply chain "controller" will dictate to member firms how much to invest in infrastructure to maximize centralized profits. Subsequently, we shall discuss inducement for firms to cooperate and thereby reach a sustainable centralized investment strategy in the long-run. In particular an example will be treated in detail. The centralized supply chain investment problem is formulated as a control problem:

$$\operatorname*{Max}_{L_j, I_j} \int_0^\infty \sum_{j=1}^{n} \mathrm{e}^{-r_j t}[p_j(t)f(K(t), L_j(t)) - c_j(t)L_j(t)$$
$$- C_{Ij}((1-\theta)I_j(t))]\mathrm{d}t$$
$$\text{subject to (1).} \qquad (19)$$

whose Hamiltonian is:

$$H(t) = \sum_{j=1}^{N} \mathrm{e}^{-r_j t}[p_j(t)f(K(t), L_j(t)) - c_j(t)L_j(t)$$
$$- C_{Ij}((1-\theta)I_j(t))] + \psi(t)\left(\sum_{j=1}^{N} I_j(t) - \delta K(t)\right), \qquad (20)$$

where the co-state variable $\psi(t)$ is determined by

$$\dot{\psi}(t) = -\sum_{j=1}^{N} \mathrm{e}^{-r_j t}p_j(t)\frac{\partial f(K(t), L_j(t))}{\partial K(t)}$$
$$+ \delta\psi(t), \quad \lim_{t\to\infty}\psi(t) = 0. \qquad (21)$$

Maximizing the Hamiltonian with respect to investments, $I_j(t)$, and labor, $L_j(t)$, we note that the optimal employment remains the same as that of the non-cooperative game, (5) and (6), while the investment strategy now depends on a single co-state variable:

$$I_j(t) = \begin{cases} i_j(t), & \text{if } \psi(t) \geqslant \frac{\partial C_{Ij}(0)}{\partial I_j}\mathrm{e}^{-r_j t}, \\ 0, & \text{otherwise}, \end{cases} \qquad (22)$$

where $i_j(t)$ is determined by

$$\psi(t) = \frac{\partial C_{Ij}((1-\theta)i_j(t))}{\partial I_j}\mathrm{e}^{-r_j t}. \qquad (23)$$

This change implies that the optimal co-investment in the centralized chain is different from that for the corresponding decentralized chain. As with Lemma 1, we outline first the general case, summarized by Lemma 4 below.

Lemma 4. *If $a(t)$, $b_j(t)$ and $i_j(t)$, $j=1,\dots,N$ satisfy the following system of equations for $t \geqslant 0$,*

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = -\delta a(t) + \sum_{j=1}^{N} i_j(t), \quad a(0) = K_0,$$

$$b(t) = \frac{\partial C_{Ij}((1-\theta)i_j(t))}{\partial I_j} e^{-r_j t},$$

$$\frac{\partial f(a(t), l_j(t))}{\partial L_j} = \frac{c_j(t)}{p_j(t)}, \quad j = 1, \dots, N,$$

$$\dot{b}(t) = -\sum_{j=1}^{N} e^{-r_j t} p_j(t) \frac{\partial f(a(t), l_j(t))}{\partial a(t)} + \delta b(t), \quad \lim_{t\to\infty} b(t) = 0,$$

then, the strategy pair $\{Ij^*(t) = i_j(t),\ j=1,\dots,N\}$ and $\{Lj^*(t) = l_j(t), j = 1,\dots,N\}$, $t \geqslant 0$ is optimal for the centralized supply chain problem (19) and (1).

Proof. See Appendix. □

The centralized optimal solution is, of course, more profitable. As a result, a centralized investment strategy may be desirable but it may also be difficult to implement. A sustainable cooperative solution where the profits of centralization "are appropriately" distributed among firms would provide a self-enforceable procedure that allows the implementation of such a solution. To attain such a self-enforced cooperation we consider a special static-investments case. Our results are summarized by Lemma 5 below and by some examples.

Lemma 5. *Let $\sum_{i=1}^{N} \mathrm{e}^{-r_i + r_j} p_j(t) \frac{\partial f(\widehat{K}, \hat{l}_j(t))}{\partial K} \Big/ \frac{\partial C_{Ij}((1-\theta)\hat{i}_j)}{\partial I_j} = \vartheta_j$, and $\widehat{K}$, $\hat{i}_j$, $\hat{l}_j(t)$, $j = 1,\dots,N$, $t \geqslant 0$, be such that:*

$$\frac{\partial C_{Ij}((1-\theta)\hat{i}_j)}{\partial I_j}(r_j+\delta)\,\mathrm{e}^{-r_j t} = \sum_{i=1}^{N} \mathrm{e}^{-r_i t} p_i(t) \frac{\partial f(\widehat{K}, \hat{l}_i(t))}{\partial K},$$

$$\sum_{j=1}^{N} \hat{i}_j = \delta \widehat{K}, \quad \frac{\partial f(\widehat{K}, \hat{l}_j(t))}{\partial L_j} = \frac{c_j(t)}{p_j(t)}, \quad j = 1, \dots, N.$$

If $K_0 = \widehat{K}$, then there exists the pair of strategy sets: static investment $\{Ij^*(t) = \hat{i}_j,\ j = 1,\dots,N\}$ and employment $\{Lj^*(t) = \hat{l}_j(t), j = 1,\dots,N\}$, $t \geqslant 0$ which is optimal for the centralized supply chain problem (1) and (19).

Proof. See Appendix. □

Setting $r_j = r$ for $j = 1,\dots,N$ for comparing Lemmas 2 and 5, we observe that the only difference is that instead of the equation,

$$\frac{\partial C_{Ij}((1-\theta)\bar{i}_j)}{\partial I_j}(r+\delta) = p_j(t) \frac{\partial f(\overline{K}, l_j(t))}{\partial K}, \tag{24}$$

determined by Lemma 2, the following equation results from Lemma 5,

$$\frac{\partial C_{Ij}((1-\theta)\hat{i}_j)}{\partial I_j}(r+\delta) = \sum_{i=1}^{N} p_i(t) \frac{\partial f(\widehat{K}, \hat{l}_i(t))}{\partial K}. \tag{25}$$

Consequently, we can conclude that the difference between a centralized and a decentralized supply chain is that in a centralized supply chain, investments by each firm are proportional to the total supply chain production rate per capital unit. On the other hand, in a decentralized supply chain, investments by firms are only proportional to firms' production rate per capital unit. Thus, the more firms cooperate and invest proportionally to the overall supply chain production rate, the closer the decentralized investment strategy is to the centralized one. The incentive for such cooperation is evident: Firms should share in the total supply chain profits such that their profit will increase comparatively to the non-cooperative (decentralized) solution. An example to this effect is considered next.

Example 4. Consider again Example 2 with $f(K, L_j) = aK^{\alpha}L_j^{\beta}$, with $\alpha + \beta = 1$, $C_{Ij}(I) = c_{Ij}(1-\theta)I_j^2$, $r_j = r$ for $j = 1,\dots,N$, $K_0 = \overline{K}$, and $\omega_j = \left[\frac{p_j(t)}{c_j^{\beta}(t)}\right]^{\frac{1}{1-\beta}}$.

Using $\frac{\partial f(\widehat{K}, \hat{l}_j(t))}{\partial L_j} = \frac{c_j(t)}{p_j(t)}$ of Lemma 5, we have $L_j(t) = \hat{l}_j(t) = \left[\frac{1}{a\beta}\frac{c_j(t)}{p_j(t)\widehat{K}^{\alpha}}\right]^{\frac{1}{\beta-1}}$ which is identical to the labor condition found in Example 2. Next, taking into account (25) and $\sum_{j=1}^{N} \hat{i}_j = \delta\widehat{K}$ of Lemma 5, we obtain the algebraic system of $N+1$ equations in $N+1$ unknowns, $\hat{i}_j$, $j = 1,\dots,N$ and $\widehat{K}$:

$$\sum_{i=1}^{N} [w_i] \frac{\alpha}{\beta} [a\beta]^{\frac{1}{1-\beta}} \widehat{K}^{\frac{\alpha}{1-\beta}-1} - 2c_{Ij}(1-\theta)(r+\delta)\hat{i}_j = 0,$$

$$\sum_{j=1}^{N} \hat{i}_j = \delta \widehat{K}.$$

Summing all equations, we have:

$$\sum_{j=1}^{N} \frac{1}{c_{Ij}} \sum_{j=1}^{N} [w_j] \frac{\alpha}{\beta} [a\beta]^{\frac{1}{1-\beta}} \widehat{K}^{\frac{\alpha}{1-\beta}-1} - 2(1-\theta)(r+\delta)\delta\widehat{K} = 0,$$

which with respect to $\alpha + \beta = 1$ results in,

$$\widehat{K} = \frac{\sum_{j=1}^{N} \frac{1}{c_{Ij}} \sum_{j=1}^{N} [w_j] \frac{\alpha}{\beta} [a\beta]^{\frac{1}{1-\beta}}}{2(1-\theta)(r+\delta)\delta} \quad \text{and}$$

$$\hat{i}_j = \frac{\frac{\alpha}{\beta}[a\beta]^{\frac{1}{1-\beta}} \sum_{i=1}^{N} [w_i]}{2c_{Ij}(1-\theta)(r+\delta)}, \quad j = 1, \dots, N.$$

Note in this case that subsidizing investments in a centralized supply chain can provide the same results (or better) as those obtained for a decentralized supply chain, i.e., $\frac{\partial \hat{i}_j}{\partial\theta} > 0$, $\frac{\partial^2 \hat{i}_j}{\partial\theta^2} > 0$ and $\frac{\partial K}{\partial\theta} > 0$. Comparing the result of Example 4 with that of Example 2, we observe that if $c_{Ij} = c_I$ for $j = 1,\dots,N$, the profit of the centralized supply chain is due to the fact that the optimal centralized chain capital $\widehat{K}$ increases N times and employment increases $N^{\frac{1}{\beta-1}+(\beta-1)}$

times compared to the decentralized solution. This increase is provided by higher investment by each firm, j, in the centralized chain which is now proportional to the total weighted ratio of the price index and the labor cost, $\sum_{i=1}^{N}[w_i] = \sum_{j=1}^{N}\left[\frac{p_j(t)}{c_j^{\beta}(t)}\right]^{\frac{1}{1-\beta}}$ over all firms, rather than to the individual ratio for each firm, j, $\omega_j = \left[\frac{p_j(t)}{c_j^{\beta}(t)}\right]^{\frac{1}{1-\beta}}$. This however does not guarantee that if the firms decide to cooperate by investing as required by the centralized solution, then all firms will benefit individually without a reallocation of the overall supply chain profits.

7. The Stackelberg game

So far we assumed that all firms operate under transparent symmetrical information and make a decision simultaneously. When there are information and power asymmetries, the firms' propensity to co-invest in supply chain infrastructure may be affected. In this section we shall demonstrate that such asymmetries do not matter. In other words, firms will follow a strategy similar to that of the Nash equilibrium strategy. Explicitly, we shall consider the implications to a co-investment strategy if there are leaders and followers (and firms' moves are sequential). In such a case, we can rank and number firms in a given order expressing their leader-follower relationships. Let $j=1$ denote the main leader; $j=2$ a follower to firm 1 but leader to firm $j=3$ and so on. It turns out, that the Stackelberg strategy applied to consecutive subsets of firms will result in an equilibrium identical to that obtained in case of a non-cooperating (Nash solution) supply chain. This observation implies that it does not matter who is the leader and who is the follower when it comes to developing the co-dependent supply chain infrastructure as shown in the following theorem. This characteristic of our solution explains to a large extent the robustness and the proliferation of supply chains.

Theorem 3. *If the pair of dynamic strategy sets* $\{Ij^s(t) = i_j^*(t), j = 1,\ldots,N\}$ *and* $\{Lj^s(t) = l_j^*(t), j = 1,\ldots,N\}$, $t \geqslant 0$ *is a Stackelberg equilibrium in the supply chain investment and labor force game so that each firm j is the leader of the set of firms* $j+1, j+2,\ldots,N$, *then this pair is a Nash equilibrium as well for the same game.*

Proof. See Appendix. □

Consider conditions of Example 1 for $N=3$, but solve it using the logic of Theorem 3 (with the Stackelberg approach).

Example 5. Let the strategies for leaders $j=1,2$ be given by $i_1(t) = i_1^*(t)$, $i_2(t) = i_2^*(t)$, $l_1(t) = i_1^*(t)$ and $l_2(t) = i_2^*(t)$. Then using (6) the optimal response in terms of employment of the lowest rank follower, $j=3$, is defined by $\frac{\partial f(K(t), l_3(t))}{\partial L_3} = \frac{c_3(t)}{p_3(t)}$, that is, $L_3(t) = l_3(t) = \left[\frac{1}{a\beta}\frac{c_3(t)}{p_3(t)K^{\alpha}(t)}\right]^{\frac{1}{\beta-1}}$.

Now, let also (4), $\alpha+\beta=1$ and $\omega_j(t) = \left[\frac{p_j(t)}{c_j^{\beta}(t)}\right]^{\frac{1}{1-\beta}} = \mathrm{e}^{\varepsilon t}$, then we have:

$$\dot{b}_3(t) = -\mathrm{e}^{-(r_3-\varepsilon)t}\frac{\alpha}{\beta}[a\beta]^{\frac{1}{1-\beta}} + \delta b_3(t), \quad \text{and thus}$$

$$b_3(t) = \frac{\mathrm{e}^{-(r_3-\varepsilon)t}}{r_3-\varepsilon+\delta}\frac{\alpha}{\beta}[a\beta]^{\frac{1}{1-\beta}}.$$

Solving $b_3(t) = \frac{\partial C_{Ij}((1-\theta)i_3(t))}{\partial I_3}\mathrm{e}^{-r_3 t}$ in $i_3(t)$ we find the optimal (investment) response of a firm, $j=3$ is:

$$I_3(t) = i_3(t) = \frac{\mathrm{e}^{\varepsilon t}}{r_3-\varepsilon+\delta}\frac{\alpha}{2(1-\theta)c_{I3}\beta}[a\beta]^{\frac{1}{1-\beta}},$$

while according to (1),

$$\dot{a}(t) = -\delta a(t) + i_3(t) + \sum_{j=1}^{2} i_j^*(t).$$

Given the optimal response of a follower $j=3$, we can now define a Stackelberg equilibrium for the firm-leader, $j=2$. Thus, using the same arguments as above, we have:

$$L_2(t) = l_2(t) = \left[\frac{1}{a\beta}\frac{c_2(t)}{p_2(t)K^{\alpha}(t)}\right]^{\frac{1}{\beta-1}},$$

$$b_2(t) = \frac{\mathrm{e}^{-(r_2-\varepsilon)t}}{r_2-\varepsilon+\delta}\frac{\alpha}{\beta}[a\beta]^{\frac{1}{1-\beta}}.$$

$$I_2(t) = i_2(t) = \frac{\mathrm{e}^{\varepsilon t}}{r_2-\varepsilon+\delta}\frac{\alpha}{2(1-\theta)c_{I2}\beta}[a\beta]^{\frac{1}{1-\beta}}.$$

Note, that the optimal solution for $j=2$ is independent of the follower's, $j=3$, optimal response except for the mutual capital (infrastructure), where substitution results in:

$$\dot{a}(t) = -\delta a(t) + i_3(t) + i_2(t) + i_1^*.$$

Moreover, it is easy to observe, that if $j=2$ is the only leader (i.e., $N=2$), then the equilibrium reached is identical to that found in Example 1 for $N=2$. This implies that, a Stackelberg equilibrium is in this case identical to a Nash equilibrium when $N=2$ and thereby it does not matter whether both players do simultaneous moves or sequential moves. If this is true, we can consider firms $j=2$ and $j=3$ making simultaneous moves. Thus, if $N=3$, then evidently there is no better response for both $j=3$ and $j=2$ (which are now the concurrent followers), than the above optimal solution found for arbitrary leader's, $j=1$, strategies i_1^*, l_1^*. Consequently, to find an equilibrium for our leader, $j=1$, we substitute the optimal response of firms $j=2,3$ and find:

$$L_1(t) = l_1(t) = \left[\frac{1}{a\beta}\frac{c_1(t)}{p_1(t)K^{\alpha}(t)}\right]^{\frac{1}{\beta-1}},$$

$$b_1(t) = \frac{\mathrm{e}^{-(r_1-\varepsilon)t}}{r_1-\varepsilon+\delta}\frac{\alpha}{\beta}[a\beta]^{\frac{1}{1-\beta}},$$

$$I_1(t) = i_1(t) = \frac{\mathrm{e}^{\varepsilon t}}{r_1-\varepsilon+\delta}\frac{\alpha}{2(1-\theta)c_{I1}\beta}[a\beta]^{\frac{1}{1-\beta}},$$

which again affects only the overall supply chain infrastructure capital:

$$\dot{a}(t) = -\delta a(t) + \sum_{j=1}^{3} i_j(t) \quad \text{or}$$

$$\dot{a}(t) = -\delta a(t) + \sum_{j=1}^{3} \frac{e^{\varepsilon t}}{r_j - \varepsilon + \delta} \frac{\alpha}{2(1-\theta)c_{Ij}\beta} [a\beta]^{\frac{1}{1-\beta}}.$$

The solution of this differential equation has been found in Example 1, which implies again that it does not matter whether firms' moves are simultaneous (as in Example 1) or sequential. For instance, if the leader, $j = 1$, makes a move, then both followers $j = 2$ and $j = 3$ would respond simultaneously, or firms $j = 1$ and $j = 2$ will make their move (first 1 and then 2).

8. Conclusion

Co-Investment in a supply chain infrastructure is a necessity, without which a supply chain cannot be sustainable. While macroeconomic and theoretical issues underpinning infrastructure investments have received significant attention, logistic infrastructure investment has received little consideration. This paper has focused some attention to this problem and provided a number of solutions and examples based on the solution of differential games which recognize the need for support and incentives for cooperation solutions. The effect of such support was specified in our analysis by the parameter theta which has the net effect of reducing the investment cost by supply chain member firms and thereby augmenting the supply chain pooled capital. In all cases we have pointed out to a growth of capital and co-investment when investment subsidies increase.

In addition, we have shown that if firms in a supply chain differ in their basic parameters, then firms' investments not only differ and change over time, but their co-investment shares in the overall infrastructure capital may diverge over time as well. To overcome such situations, a number of compensating measures may be followed.

A first approach, when firms are non-cooperating, emanates from Lemmas 1 and 2. In such a case, the supply chain attains asymptotically a stationary investment equilibrium (described in Lemma 2), which is characterized by steady co-investment shares. To efficiently implement this dynamic strategy, we derive a feedback Nash equilibrium (Theorem 1), which allows us to determine investments as a function of the current infrastructure capital and employment level. An alternative and second approach is based on either a short- or a long-run cooperation. We show that even if competing firms are able to cooperate only in a short-run, there exist a one-time investment such that the firms can reach a stationary Nash equilibrium in no-time and stay there infinitely long (Theorem 2). We have also shown that, the firms' propensity to co-invest in supply chain infrastructure is not affected when there are information and power asymmetries (Theorem 3), and investment in infrastructure differs when the supply chain has a centralized (centrally controlled) organizational mode. Since in this latter mode, the optimal solution yields a greater overall profit, a long-run cooperation is advantageous and can be maintained if the supply chain redistributes excess profits to member firms to sustain the supply chain viability. We find that in a centralized supply chain, investments by each firm are proportional to the total supply chain production rate per capital unit. On the other hand, in a decentralized supply chain, investments by firms are only proportional to firms' production rate per capital unit. Thus, the more firms cooperate and invest proportionally to the overall supply chain production rate, the closer the decentralized investment strategy is becoming to that of the centralized one. If the centralized supply chain profits are high it is also possible that one of the members of the supply chain will find it advantageous to takeover all firms and thereby benefit from the global economy of scale that the supply chain provides.

The importance of our approach to "cooperating, sharing and sustainability" provides additional opportunities for extensions and further research in the intricate problems we face when a supply chain sustainable co-investment strategy has to be defined and managed. Furthermore, to maintain the "control tractability" of our problem, a number of important factors and issues were neglected. For example, investments can be financed not only by subsidizing, but by tax collection and borrowing as well. While in some cases, borrowing may be attractive in the short-run it can also be non-sustainable. For this reason, issues associated to how much and at what price to borrow are clearly worth considering. We have also neglected the potential to finance investment in infrastructure through the market mechanism. This is of course an important issue to reckon with and provides an opportunity for future research.

Appendix

Proof of Lemma 1. The proof is straightforward. Solutions $a_j(t)$, $b_j(t)$ and $i_j(t)$, $j = 1,\dots,N$ evidently satisfy state (1) and co-state (4) equations. Furthermore, the strategies $l_j(t)$, $i_j(t)$, $j = 1,\dots,N$ are optimal if conditions (5) and (7) hold for all $j = 1,\dots,N$. That is, N Hamiltonians are maximized simultaneously, if:

$$\frac{\partial f(a(t), l_j(0))}{\partial L_j} \leqslant \frac{c_j(t)}{p_j(t)} \quad \text{and} b_j(t) \geqslant \frac{\partial C_{Ij}(0)}{\partial I_j} e^{-r_j t},$$
$$j = 1,\dots,N,$$

which is evidently always met since $\frac{\partial f}{\partial L_j} > 0$ for $L \neq 0$, $\frac{\partial f(K,0)}{\partial L_j} = 0$, $\frac{\partial C_{Ij}}{\partial I_j} \geqslant 0$ and thereby, $b_j(t) = \frac{\partial C_{Ij}((1-\theta)i_j(t))}{\partial I_j} e^{-r_j t} \geqslant \frac{\partial C_{Ij}(0)}{\partial I_j} e^{-r_j t}$, as stated in this lemma. □

Proof of Lemma 2. Consider the solution for the state and co-state variables, which is characterized by a constant level of capital $K(t) = \overline{K}$:

$$\dot{\psi}_j(t) = -\mathrm{e}^{-r_j t} p_j(t) \frac{\partial f(K(t), l_j(t))}{\partial K(t)} + \delta\psi_j(t), \quad \lim_{t\to\infty} \psi_j(t) = 0,$$
$$\frac{\mathrm{d}K(t)}{\mathrm{d}t} = -\delta K(t) + \sum_{j=1}^{N} I_j(t) = 0.$$

This solution is optimal if $l_j(t)$ and $i_j(t)$ from (5) and (7),

$$\frac{\partial f(K(t), l_j(t))}{\partial L_j} - \frac{c_j(t)}{p_j(t)} = 0,$$
$$\psi_j(t) = \frac{\partial C_{Ij}((1-\theta)i_j(t))}{\partial I_j}\,\mathrm{e}^{-r_j t},$$

are non-negative which is evidently true as $\frac{\partial f}{\partial L_j} \geqslant 0$ for $L \neq 0$, $\frac{\partial f(K,0)}{\partial L_j} = 0$, $\frac{\partial C_{Ij}}{\partial I_j} \geqslant 0$ and

$$\psi_j(t) = \frac{\partial C_{Ij}((1-\theta)i_j(t))}{\partial I_j}\,\mathrm{e}^{-r_j t} \geqslant \frac{\partial C_{Ij}(0)}{\partial I_j}\,\mathrm{e}^{-r_j t}.$$

Differentiating $\psi_j(t) = \frac{\partial C_{Ij}((1-\theta)i_j(t))}{\partial I_j}\,\mathrm{e}^{-r_j t}$and substituting the co-state equation, we find that:

$$\dot{\psi}_j(t) = -\frac{\partial C_{Ij}((1-\theta)\bar{i}_j)}{\partial I_j} r_j \mathrm{e}^{-r_j t}$$
$$= -\mathrm{e}^{-r_j t} p_j(t) \frac{\partial f(\overline{K}, l_j(t))}{\partial K} + \delta \frac{\partial C_{Ij}((1-\theta)\bar{i}_j)}{\partial I_j} r_j \mathrm{e}^{-r_j t},$$

If $p_j(t)\frac{\partial f(\overline{K}, l_j(t))}{\partial K} \Big/ \frac{\partial C_{Ij}((1-\theta)\bar{i}_j)}{\partial I_j} = \vartheta_j$ is constant, then capital $\overline{K}$ and investment $\sum_{j=1}^{N} \bar{i}_j = \delta\overline{K}$ policies are constant as well, as stated in the lemma. Finally, we straightforwardly verify the boundary condition,

$$\lim_{t\to\infty} \psi_j(t) = \lim_{t\to\infty} \frac{\partial C_{Ij}((1-\theta)\bar{i}_j)}{\partial I}\,\mathrm{e}^{-r_j t} = 0. \qquad \square$$

Proof of Lemma 3. The lemma is proved by contradiction. Let us assume that the optimal solution is obtained for $t^* > 0$. Then $I_j(t)$, $i = 1,\ldots,N$ are finite (otherwise any capital can be reached in no time which contradicts $t^* > 0$) and thus $\lambda(t) \leqslant 0$. On the other hand, according to the maximum principle, if $\lambda(t) < 0$, $0 \leqslant t \leqslant t^*$, then no capital is invested and thereby $K(t)$ will never reach $\overline{K}$. Accordingly, the only case left is $\lambda(t) = 0$ for a measurable interval of time. However, condition (17) never holds during this interval, if $\lambda(t) = 0$. Thus, we get a contradiction and $t^* = 0$.

To meet the boundary condition $K(t^*) = \overline{K}$, we need:

$$K_0 + \int_0^{0+} \left[-\delta K(t) + \sum_{j=1}^{N} I_j(t) \right] \mathrm{d}t = \overline{K}.$$

Thus, $I_j(t^*) = \widehat{I}_j \Delta(t^*)$ and substituting this into the last expression we find an equation for unknowns $\widehat{I}_j$, as stated in the lemma. $\square$

Proof of Lemma 4. The proof is similar to that for Lemma 1. Solutions $a_j(t)$, $b_j(t)$ and $i_j(t)$, $j = 1,\ldots,N$ evidently satisfy state (1), co-state (21) equations and optimality conditions (5), (6), (22) and (23) if:

$$\frac{\partial f(a(t), l_j(0))}{\partial L_j} \leqslant \frac{c_j(t)}{p_j(t)} \quad \text{and} \quad b(t) \geqslant \frac{\partial C_{Ij}(0)}{\partial I_j}\,\mathrm{e}^{-r_j t},$$
$$j = 1,\ldots,N,$$

holds which is evidently always met as $\frac{\partial f}{\partial L_j} > 0$ for $L \neq 0$, $\frac{\partial f(K,0)}{\partial L_j} = 0$, $\frac{\partial C_{Ij}}{\partial I_j} \geqslant 0$ and thereby, $b(t) = \frac{\partial C_I((1-\theta)i_j(t))}{\partial I_j}\,\mathrm{e}^{-r_j t} \geqslant \frac{\partial C_I(0)}{\partial I_j}\,\mathrm{e}^{-r_j t}$, as stated in this lemma. $\square$

Proof of Lemma 5. Consider the solution for the state and co-state variables, which is characterized by a constant level of capital $K(t) = \widehat{K}$:

$$\dot{\psi}(t) = -\sum_{j=1}^{N} \mathrm{e}^{-r_j t} p_j(t) \frac{\partial f(K(t), l_j(t))}{\partial K(t)} + \delta\psi(t), \quad \lim_{t\to\infty} \psi(t) = 0,$$
$$\frac{\mathrm{d}K(t)}{\mathrm{d}t} = -\delta K(t) + \sum_{j=1}^{N} I_j(t) = 0.$$

This solution is optimal if $l_j(t)$ and $i_j(t)$ satisfy (5) and (7),

$$\frac{\partial f(K(t), l_j(t))}{\partial L_j} - \frac{c_j(t)}{p_j(t)} = 0, \quad \psi(t) = \frac{\partial C_{Ij}((1-\theta)i_j(t))}{\partial I_j}\,\mathrm{e}^{-r_j t}.$$

Differentiating $\psi(t) = \frac{\partial C_{Ij}((1-\theta)i_j(t))}{\partial I_j}\,\mathrm{e}^{-r_j t}$, we find that:

$$\dot{\psi}(t) = -\frac{\partial C_{Ij}((1-\theta)\hat{i}_j)}{\partial I_j} r_j \mathrm{e}^{-r_j t}$$
$$= -\sum_{i=1}^{N} \mathrm{e}^{-r_i t} p_i(t) \frac{\partial f(\widehat{K}, \hat{l}_i(t))}{\partial K} + \delta\psi(t), \quad j = 1,\ldots,N.$$

That is,

$$\frac{\partial C_{Ij}((1-\theta)\hat{i}_j)}{\partial I_j} (r_j + \delta)\mathrm{e}^{-r_j t} = \sum_{i=1}^{N} \mathrm{e}^{-r_i t} p_i(t) \frac{\partial f(\widehat{K}, \hat{l}_i(t))}{\partial K},$$
$$j = 1,\ldots,N.$$

If $\sum_{i=1}^{N} \mathrm{e}^{-r_i + r_j} p_j(t) \frac{\partial f(\widehat{K}, \hat{l}_i(t))}{\partial K} \Big/ \frac{\partial C_{Ij}((1-\theta)\hat{i}_j)}{\partial I_j} = \vartheta_j$ is constant then the capital and the investment policies are constant as well and are given by $\widehat{K}$ and $\sum_{j=1}^{N} \hat{i}_j = \delta\widehat{K}$, as stated in this lemma. Finally, we straightforwardly verify the boundary condition,

$$\lim_{t\to\infty} \psi(t) = \lim_{t\to\infty} \sum_{j=1}^{N} \frac{\partial C_{Ij}((1-\theta)\hat{i}_j)}{\partial I}\,\mathrm{e}^{-r_j t} = 0. \qquad \square$$

Proof of Theorem 3. Let solution for all the leaders but $j = N$ be given by $i_j(t) = i_j^*(t)$, $l_j(t) = i_j^*(t)$ for $j = 1,\ldots,N-1$. Then with respect to (5) and (7) the best response of the lowest rank follower, $j = N$, is

$$\dot{K}(t) = -\delta K(t) + i_N(t) + \sum_{j=1}^{N-1} i_j^*(t), \quad K(0) = K_0,$$

$$\psi_N(t) = \frac{\partial C_{Ij}((1-\theta)i_N(t))}{\partial I_N} \mathrm{e}^{-r_N t}, \quad \frac{\partial f(K(t), l_N(t))}{\partial L_N} = \frac{c_N(t)}{p_N(t)},$$

$$\dot{\psi}_N(t) = -\mathrm{e}^{-r_N t} p_N(t) \frac{\partial f(K(t), l_N(t))}{\partial K(t)} + \delta b_N(t),$$

$$\lim_{t\to\infty} \psi_N(t) = 0.$$

This response should then be substituted into the optimality conditions for firm $N-1$, by setting $i^*_{N-1}(t) = i_{N-1}(t)$ and $l^*_{N-1}(t) = l_{N-1}(t)$:

$$\dot{K}(t) = -\delta K(t) + i_N(t) + i_{N-1}(t) + \sum_{j=1}^{N-2} i^*_j(t),$$

$$K(0) = K_0, \quad \psi_{N-1}(t) = \frac{\partial C_{Ij}((1-\theta)i_{N-1}(t))}{\partial I_{N-1}} \mathrm{e}^{-r_{N-1} t},$$

$$\frac{\partial f(K(t), l_{N-1}(t))}{\partial L_{N-1}} = \frac{c_{N-1}(t)}{p_{N-1}(t)},$$

$$\dot{\psi}_{N-1}(t) = -\mathrm{e}^{-r_{N-1} t} p_{N-1}(t) \frac{\partial f(K(t), l_{N-1}(t))}{\partial K(t)} + \delta b_{N-1}(t),$$

$$\lim_{t\to\infty} \psi_{N-1}(t) = 0,$$

$$\psi_N(t) = \frac{\partial C_{Ij}((1-\theta)i_N(t))}{\partial I_N} \mathrm{e}^{-r_N t}, \quad \frac{\partial f(K(t), l_N(t))}{\partial L_N} = \frac{c_N(t)}{p_N(t)},$$

$$\dot{\psi}_N(t) = -\mathrm{e}^{-r_N t} p_N(t) \frac{\partial f(K(t), l_N(t))}{\partial K(t)} + \delta b_N(t),$$

$$\lim_{t\to\infty} \psi_N(t) = 0.$$

It is easy to observe now, that by setting $b_j(t) = \psi_j(t)$, $a_j(t) = K_j(t)$ and continuing the substitution for $j = N-2$, $N-3$ and so on, at $j = 1$, we finally obtain the system of equations which is identical to that stated in Lemma 1 for the most general supply chain investment and labor force game. □

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