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Decision Support

Risk-averse order policies with random prices in complete market and retailers' private information

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ABSTRACT

We consider a retailer who orders products before the price for them becomes known. The price is an outcome of perfect competition in a complete market. Since the demand is price sensitive, the uncertainty in prices induces uncertain profits and associated risks. In this paper we show that if the retailer is risk averse and, as a result, selects a utility function of profit to maximize, then his subjective assessment of future prices is affected by the risk attitude. This, in turn, introduces a bias in retailer's ordering policies. By considering coordinated pricing and ordering policies we derive a relationship between risk aversion, retailer's subjective (private) assessment and the market implied, risk neutral forecast. This relationship and the induced bias are then illustrated for two typical operations management strategies which involve either inventory considerations or promotions avoiding accumulation of stocks.

1. Introduction

This is concerned with the order policy of risk averse retailers, confronted with future random prices and an uncertain demand, (a function of prices) when prices are determined by markets in equilibrium. Namely, when market prices are unique, there is no arbitrage and thereby markets are said to be complete. In such a case, the order policy consists in ordering a certain quantity of product from a supplier at time $t = 0$ based on a private estimate of future prices and demands by the retailer while market prices are determined by competitive forces. These market forces are implied in a risk neutral distribution which is used to characterize the manner in which prices are determined today based on a "market belief" regarding prices at time $t = T$. Such an approach is common in (fundamental) financial pricing (for example, see Tapiero, 2005) and has attracted relatively little attention in dealing with operational management problems (for exceptions, see Kogan and Tapiero, 2007, Chapter 7; Tapiero, submitted for publication as well as Oum et al., 2006). Its particular advantage however is that it provides a means to relate operational policies to market prices and thereby to select order and inventory policies that are far more sensitive to markets response (or at least to the market belief regarding future conditions).

To determine a market sensitive retailer optimal order policy, we first assume that the retailer has a risk preference for money characterized by a utility function as well as a private (subjective) assessment of future prices at $t = T$. Maximization of the future expected utility with respect to the terminal consequences of the order policy (and of course the price and demand uncertainty) as well as the market estimate on implied future prices will yield the optimal utility for the retailer at $t = T$. Practically, this corresponds to the retailer providing a private forecast of future prices, denoted by the probability distribution $f_P(\cdot)$ while the implied (believed to be) future prices, are denoted by the risk neutral probability distribution $f_{RN}(\cdot)$. As we shall show in this paper, for a rational retailer, there is a unique relationship between the retailer index of risk aversion, his private distribution and the risk neutral (market) probability distribution. This is given by a proposition, which is proved in the appendix to the paper. Although this approach is known in financial engineering (see Jackworth, 1999, 2000; Ait-Sahalia and Lo, 1998; Alonzo, 2005; Munier and Tapiero, 2008; Perignon and Villa, 2002; Tapiero, 2007), our extension to operational management is two fold. First, we show that the extension can be formulated and studied as an optimal control problem (based on the Maximum Principle). Secondly, we integrate such an option within an optimal (order) policy selection. In addition, we consider a number of traditional inventory control examples and show how our results and our approach differ significantly and perhaps usefully from the traditional approaches. Finally, we also extend our analysis to include a jointly random demand function and future prices. Because of the tractability issues such a problem will be pursued in a subsequent and companion paper. Notwithstanding the assumptions made in this paper we highlight

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some of the difference between the private and risk neutral probability distributions and point out to a bias which affects the choice of the optimal order quantity as a function of the retailer risk attitudes. While in Section 2 such a bias is determined; in Section 3, two typical inventory related applications are solved. In these examples, we derive specific relationships between the order policy, the retailer risk attitudes, the private and the risk neutral (market) probability distribution of prices. For example, we shall show that when demands depend linearly on price (Bertrand's model), given one of the probability functions, say, risk neutral, the other, private function, may involve up to two parts—one biased, and the other which is not affected by the retailer risk attitude. Further, we also introduce a policy of price breaks as a means to eliminate ex-post inventories. In such circumstances it is shown that the order quantity decreases (as one would expect from a decision maker with risk averse attitude). Finally, Section 4 considers the important extension to random demands, both in terms of the random market prices and some independent random parameters.

2. Retailer risk aversion, private and the risk neutral distribution

Let π be the future (random) market price of a product at time $t = 0$ and observable at time $t = T$ as a realization p. Assume also that the demand S for the product at $t = T$, is determined by Bertrand's model, $S = S(P(p))$, $\frac{dS}{dP} < 0$, a function of price $P(p)$ that the retailer chooses at $t = T$ with respect to the realized market price, p. Accordingly, denote the sales and the costs associated with such business as $Q(q,S(P(p)))$ and $C(q,S(P(p)))$, respectively, where q is the quantity ordered by the retailer. Assume that the retailer chooses the final product price $P(p)$ for each possible realization p at $t = T$ so that his expected utility of profits $U[(P(p) - w)Q(q, S(P(p))) - C(q,S(P(p)))]$ is maximized for any quantity q ordered from a supplier at a price w at time $t = 0$. In the traditional approach practiced in operations management, the retailer will also seek to optimize the order quantity q (by maximizing the retailer expected utility of profit). Such a situation will result in the objective function $J(P,q)$ to be maximized with respect to the two decision variables as stated in (1) and (2) below:

$$\max_{P \geqslant 0, q \geqslant 0} J(P,q) = \max_{P \geqslant 0, q \geqslant 0} E_P(\Omega(q, P(\pi))), \tag{1}$$

where

$$\Omega(q, P(\pi)) = (P(\pi) - w)Q(q, S(P(\pi))) - C(q, S(P(\pi))) \tag{2}$$

and subject to the complete market profit estimate

$$\Pi = \frac{1}{(1+R_f)^T} \int_0^\infty \Omega(q, P(p)) f_{RN}(p) dp, \tag{3}$$

where Π is the amount of money the retailer invests in the operations.

An explicit expression for (1)–(3) in terms of the retailer's private estimate of the future probability distribution $f_P(p)$ of prices is

$$\max_{P \geqslant 0, q \geqslant 0} J(P,q) = \max_{P \geqslant 0, q \geqslant 0} \int_0^\infty U[\Omega(q, P(p))] f_P(p) dp \quad \text{Subject to}(3). \tag{4}$$

Of course, the retailer's forecast $f_P(p)$ may differ from that implied in the beliefs regarding future prices and denoted by $f_{RN}(p)$.

The problem (4) and (2) is a Variational (dynamic) problem in $P(p)$ and a static problem in the order quantity q. We first apply the maximum principle to derive the optimality conditions for the Variational problem with respect to $P(p)$. Subsequently, an optimal order q is determined. A simultaneous solution to these optimality conditions solves problem (4) subject to (2).

2.1. Optimization with respect to the price $P(p)$ to be set by the retailer

The solution of this problem in $P(p)$ proved in the appendix results in the following proposition.

Proposition 1. *Let functions $\Omega(q,P(p))$, $f_{RN}(p)$, $f_P(p)$ be differentiable almost everywhere with respect to prices and let $r_u[\Omega(q,P(p))]$ be the retailer's index (Arrow-Pratt) of absolute risk aversion. Then*

$$\frac{\partial \Omega(q, P(p))}{\partial p} r_u[\Omega(q, P(p))] = \frac{d}{dp}\left(\ln \frac{f_P(p)}{f_{RN}(p)}\right), \quad p \geqslant 0. \tag{5}$$

Proof . See Appendix 1.

There are two immediate observations emanating from (5). First, if the retailer is risk neutral, $r_u(\cdot) = 0$, then taking into account that $\int_0^\infty f_P(p)dp = \int_0^\infty f_{RN}(p)dp = 1$, we find that $f_{RN}(p) = f_P(p)$. *That is, a risk neutral retailer does not introduce any bias in his forecast of prices*. Second, once a subjective (private) forecast of prices has been made by a risk averse retailer, then this forecast introduces a bias that affects decision making each time the private forecast is employed, even if the utility is no longer used in selecting an optimal policy.

Note, that Eq. (5) has a general form in terms of possible pricing policies. If we adopt the assumption that the competition is perfect (no arbitrage), then in equilibrium the retailer cannot set a price $P(p)$ higher than the market price p. However, through a promotional strategy, it is possible to select a smaller price which cannot exceed the market price, i.e., $P(p) \leqslant p$. On the other hand, if no promotion strategy is adopted during the time interval $[0,T]$, then in a complete market, $P(p) = p$, uniquely determines the retailer's wealth and Eq. (5), taking the following form:

$$\frac{\partial \Omega(q, p)}{\partial p} r_u[\Omega(q, p)] = \frac{d}{dp}\left(\ln \frac{f_P(p)}{f_{RN}(p)}\right), \quad p \geqslant 0. \tag{6}$$

2.2. Optimization with respect to the order quantity q

We consider next the optimization of the order quantity q. Assume that $Q(q,S(P))$ and $-C(q,S(P))$ are concave in q (as it is typically the case for news-vendor type of problems), Eq. (3) may have then only a few feasible solutions. If the complete market constraint (3) has a unique solution with respect to $q > 0$, then such a solution is optimal. Otherwise, the objective function (4) should be maximized with respect to all feasible solutions of equation (3) in q. This can be accomplished by the Lagrange multipliers' method. Since the number of feasible solutions is limited, then by a simple verification of the objective function value, an optimal policy can be determined. Let $q_1 \geqslant 0$ and $q_2 \geqslant 0$ satisfy (3). Consequently, employing again the condition for no arbitrage and no promotion by the retailer, the objective function will depend only on the order quantity, either $J(q_1)$ or $J(q_2)$. Thus, the optimal solution q^* is found by

$$q^* = \begin{cases} q_1 & \text{if} \quad J(q_1) > J(q_2), \\ q_2 & \text{otherwise}, \end{cases} \tag{7}$$

where Eq. (3) takes the following form:

$$\Pi = \frac{1}{(1+R_f)^T} \int_0^\infty \Omega(q, p) f_{RN}(p) dp. \tag{8}$$

This implies that a simultaneous solution of equations (6) and (7) along with condition (8) provides the optimal order quantity q^* and the risk neutral probability $f_{RN}(p)$.

Using these results, we note that the effect of the retailer's bias on the optimal order quantity is directly observed. If the retailer employs his private (subjective) estimate, $f_P(p)$, instead of the objective, $f_{RN}(p)$, then an optimal order, say, q_P, can be determined, then Eq. (8) transforms into

$$\Pi = \frac{1}{(1+R_f)^{\mathrm{T}}} \int_0^{\infty} \Omega(q_{\mathrm{P}}, P(p)) f_{\mathrm{P}}(p) \mathrm{d}p, \tag{9}$$

and the difference $(q_{\mathrm{P}} - q^*)$ would characterize an error due to the retailer's risk induced bias. Specifically, Eq. (5) determines $f_{\mathrm{RN}}(p)$ as an implicit function, say $F(\cdot)$, of the optimal order quantity $q = q^*$ and $f_{\mathrm{P}}(p)$ for each price p, i.e., $f_{\mathrm{RN}}(p) = F(f_{\mathrm{P}}(p), q^*, p)$. Therefore, by inserting this relationship into (8), we have

$$\Pi = \frac{1}{(1+R_f)^{\mathrm{T}}} \int_0^{\infty} [\Omega(q^*, p)] F(f_{\mathrm{P}}(p), q^*, p) \mathrm{d}p. \tag{10}$$

Thus, given a retailer's forecast, $f_{\mathrm{P}}(p)$, and thereby the implied optimal order policy q_{P}, the effect of the retailer's risk attitude on the forecast and order quantity is assessed by solving Eq. (10) in q^* or equivalently by solving simultaneously (7) and (8) in q^* and $f_{\mathrm{RN}}(p)$. While a general solution might be difficult to obtain, specific examples can be used to illustrate these effects explicitly, as is shown in the next section.

3. Applications

We shall consider absolute linear inventory costs with h^+ and h^- denoting the unit cost of an inventory surplus and shortage, respectively, to be consistent with the traditional approach to inventory costing. Such an approach presumes that inventory costs are defined initially when the order is made. An appropriate alternative to this approach would consist in letting the inventory costs be proportional to the uncertain price however.

Then sales are given by, $Q(q,S(\pi)) = \min\{q, S(\pi)\} = \min\{q, a - b\pi\}$ while the inventory related cost is

$$C(q,S(\pi)) = h^+ \max\{q - S(\pi), 0\} + h^- \max\{S(\pi) - q, 0\}. \tag{11}$$

Therefore (see Fig. 1)

$$\frac{\partial Q}{\partial \pi} = \begin{cases} 0 & \text{if } 0 \leqslant \pi < \frac{a-q}{b} \\ -b & \text{if } \frac{a-q}{b} < \pi \leqslant \frac{a}{b}; \\ z \in [-b, 0] & \text{if } \pi = \frac{a-q}{b} \end{cases}$$
$$\frac{\partial S}{\partial \pi} = \begin{cases} bh^+ & \text{if } \frac{a-q}{b} < \pi \leqslant \frac{a}{b} \\ -bh^- & \text{if } 0 \leqslant \pi < \frac{a-q}{b} \\ \bar{z} \in [-bh^-, bh^+] & \text{if } \pi = \frac{a-q}{b}. \end{cases} \tag{12}$$

Accordingly, and recalling our notation for $\Omega(q,p)$, Eq. (6) assumes the following form:

$$(q + bh^-) r_u [(p - w)q - h^-(a - bp - q)] = \frac{\mathrm{d}}{\mathrm{d}p}\left(\ln \frac{f_{\mathrm{P}}(p)}{f_{\mathrm{RN}}(p)}\right) \text{ for } 0 \leqslant p < \frac{a-q}{b}, \tag{13a}$$

$$(a - bp - b(p - w) - bh^+) r_u [(p - w)(a - bp) - h^+(q - a + bp)] = \frac{\mathrm{d}}{\mathrm{d}p}\left(\ln \frac{f_{\mathrm{P}}(p)}{f_{\mathrm{RN}}(p)}\right), \quad \frac{a-q}{b} < p \leqslant \frac{a}{b}. \tag{13b}$$

Thus, for a given order, q, and private probability function $f_{\mathrm{P}}(p)$, the differential Eq. (13) can be solved for the risk neutral probability $f_{\mathrm{RN}}(p)$. Note that the only ambiguous point $p = \frac{a-q}{b}$ cannot affect calculation of the integral equation (9) when determining an optimal order q^*. The boundary condition for the differential equation is straightforward

$$\int_0^{\frac{a}{b}} f_{\mathrm{RN}}(p) \mathrm{d}p = 1. \tag{14}$$

Accounting for (11), Eq. (8) takes the following form:

$$\Pi = \frac{1}{(1+R_f)^{\mathrm{T}}} \left\{ \int_0^{\frac{a-q^*}{b}} [(p - w)q^* - h^-(a - bp - q^*)] f_{\mathrm{RN}}(p) \mathrm{d}p + \int_{\frac{a-q^*}{b}}^{\frac{a}{b}} [(p - w)(a - bp) - h^+(q^* - a + bp)] f_{\mathrm{RN}}(p) \mathrm{d}p \right\}. \tag{15}$$

Solving (12)–(15) simultaneously we obtain q^* and $f_{\mathrm{RN}}(p)$. Finally, Eq. (9), defined for our example yields:

$$\Pi = \frac{1}{(1+R_f)^{\mathrm{T}}} \left\{ \int_0^{\frac{a-q_P}{b}} [(p - w)q_P - h^-(a - bp - q_P)] f_{\mathrm{P}}(p) \mathrm{d}p + \int_{\frac{a-q_P}{b}}^{\frac{a}{b}} [(p - w)(a - bp) - h^+(q_P - a + bp)] f_{\mathrm{P}}(p) \mathrm{d}p \right\}, \tag{16}$$

which provides a biased order quantity, q_{P}, for a given $f_{\mathrm{P}}(p)$.

When we consider a constant absolute risk aversion (CARA), our results, of course, simplify. Let the retailer be characterized by a constant absolute risk aversion (CARA), $r_u(\cdot) = \alpha$. Then Eq. (13a) simplifies to

$$(q + bh^-)\alpha = \frac{\mathrm{d}}{\mathrm{d}p}\left(\ln \frac{f_{\mathrm{P}}(p)}{f_{\mathrm{RN}}(p)}\right) \text{ for } 0 \leqslant p < \frac{a-q}{b}. \tag{17}$$

Therefore

$$(q + bh^-)\alpha p + \beta = \ln \frac{f_{\mathrm{P}}(p)}{f_{\mathrm{RN}}(p)}, \tag{18}$$

or

$$f_{\mathrm{P}}(p) \mathrm{e}^{-[(q+bh^-)\alpha p + \beta]} = f_{\mathrm{RN}}(p) \text{ for } 0 \leqslant p < \frac{a-q}{b}, \tag{19}$$

where β is an integration constant. Similarly, Eq. (13b) simplifies to

$$(a + bw - bh^+)\alpha - 2bp\alpha = \frac{\mathrm{d}}{\mathrm{d}p}\left(\ln \frac{f_{\mathrm{P}}(p)}{f_{\mathrm{RN}}(p)}\right) \text{ for } \frac{a-q}{b} < p \leqslant \frac{a}{b}. \tag{20}$$

Thus

$$f_{\mathrm{P}}(p) \mathrm{e}^{-[(a+bw-bh^+)\alpha p - bp^2\alpha + \gamma]} = f_{\mathrm{RN}}(p) \text{ for } \frac{a-q}{b} < p \leqslant \frac{a}{b}, \tag{21}$$

where γ is an integration constant. If we require that $f_{\mathrm{RN}}(p)$ calculated at $p = \frac{a-q}{b}$ by (19) be equal to that calculated by (21), we have

$$f_{\mathrm{P}}\left(\frac{a-q}{b}\right) \mathrm{e}^{-[(a+bw-bh^+)\alpha\frac{a-q}{b} - b\left(\frac{a-q}{b}\right)^2\alpha + \gamma]} = f_{\mathrm{P}}\left(\frac{a-q}{b}\right) \mathrm{e}^{-[(q+bh^-)\alpha\frac{a-q}{b} + \beta]}. \tag{22}$$

That is

$$(a + bw - bh^+ - bh^- - q)\alpha \frac{a-q}{b} - b\left(\frac{a-q}{b}\right)^2 \alpha + \gamma = \beta. \tag{23}$$

In addition, from (14) we have

$$\int_0^{\frac{a-q}{b}} f_{\mathrm{P}}(p) \mathrm{e}^{-[(q+bh^-)\alpha p + \beta]} \mathrm{d}p + \int_{\frac{a-q}{b}}^{\frac{a}{b}} f_{\mathrm{P}}(p) \mathrm{e}^{-[(a+bw-bh^+)\alpha p - bp^2\alpha + \gamma]} \mathrm{d}p = 1. \tag{24}$$

Conditions (23) and (24) constitute a system of two equations in two unknowns γ and β. Solving this system along with (15) provides an optimal order quantity and risk neutral probability function. Inversely, given a risk neutral distribution using the same system of equations one can determine private probability distribution of the risk averse retailer. A numerical example provides an explicit solution. Specifically, say that the market belief regarding future prices (the risk neutral distribution) is normal with mean and

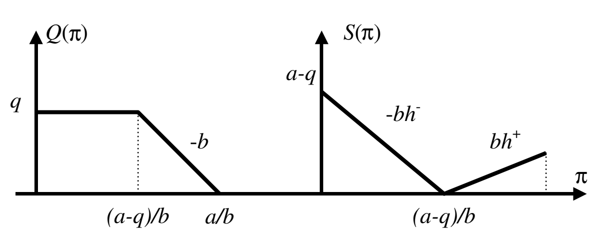

Fig. 1. Sales and inventory costs as a function of prices.

standard deviations given by $\mu := \frac{a}{2b}$, $\sigma := \frac{a}{10b}$ (which ensures that the probability of negative demands will be negligible). Assume the following parameters $R = 0.1$, $T = 1$, $\Pi = 10$, $h^+ = 0.1$, $h^- = 3$, $a = 30$, $b = 2$, $w = 2$, $\alpha = 0.1$. Then solving 23 and 24 with Maple, we find the subjective probability distribution (see Fig. 2)

$$f_P(p) = \frac{1}{3}\frac{\sqrt{2}\mathrm{e}^{\left(-\frac{2(p-15/2)^2}{9}\right)}\mathrm{e}^{(1.259186265p-11.02873120)}}{\sqrt{\pi}}. \tag{25}$$

The optimal order quantity is then $q^* = 6.591862649$ and the order quantity with respect to the retailer's bias is: $q_p = 3.269144136$.

Fig. 2 points out to retailer (privately) overestimating the probability densities of future prices, and, as a result, inducing an expectation for higher price than that implied by the risk neutral (market) distribution. In other words, under risk aversion, the order quantity decreases as a result of the retailer believing a shift in the market prices probabilities toward higher prices.

In some cases, excess inventories are met by retailers by price break promotions (that is, selling at a price lower than the market price). In particular, if the market price is higher than the retailer has expected when ordering products (and therefore demands are lower than expected), a promotion will be initiated to get rid of the stock. To resolve this problem with our approach, let the retailer order again a quantity $q < a$. If the market price at $t = T$, is π and such that $q > S(\pi) = a - b\pi$, then the retailer will reduce the price to have all stock sold out. The retail promotional price is then, $P(\pi) = \frac{a-q}{b} < \pi$. Accordingly, the retailer will incur no inventory surplus cost. Assuming for simplicity that w and the cost associated with lost sales are negligible, we have then $C(q,S(\pi)) = 0$. Further, when $q > S(\pi)$, the price is reduced to $P(\pi) = \frac{a-q}{b} < \pi$ and the retailer's profit $\Omega(q,P(p))$ is $qP(p)$. Otherwise, since there is no arbitrage, the retailer cannot increase the price (at prices higher than the market price there would be no sales), the profit is $q\pi$. Equivalently,

$$\Omega(q,P(\pi)) = qP(\pi) = \begin{cases} q\frac{a-q}{b} & \text{if } \frac{a-q}{b} < \pi \leqslant \frac{a}{b}, \\ q\pi & \text{otherwise} \end{cases} \tag{26}$$

and

$$\frac{\partial \Omega}{\partial \pi} = \begin{cases} q & \text{if } 0 \leqslant \pi < \frac{a-q}{b}, \\ 0 & \text{if } \frac{a-q}{b} < \pi \leqslant \frac{a}{b}, \\ \hat{z} \in [0,q] & \text{if } \pi = \frac{a-q}{b}. \end{cases} \tag{27}$$

Employing this time (26) and inserting it into (5), together with (27), we have

$$\begin{cases} qr_u[qp] = \frac{f_P'(p)}{f_P(p)} - \frac{f_{RN}'(p)}{f_{RN}(p)} & \text{when } 0 \leqslant p < \frac{a-q}{b}, \\ 0 = \frac{f_P'(p)}{f_P(p)} - \frac{f_{RN}'(p)}{f_{RN}(p)} & \text{when } \frac{a-q}{b} < p \leqslant \frac{a}{b}. \end{cases} \tag{28}$$

Recalling the independence of $\Omega(q,P(p))$ and p along an interval of prices (see Appendix and condition (26)), we note that in the absence of a concern for inventories, the retailer has no specific bias in the private probability distribution when $\frac{a-q}{b} < p \leqslant \frac{a}{b}$. Therefore, any function can be selected for this interval under condition that Eq. (14), which is the boundary condition for (28), holds. Finally, when taking into account (26), Eq. (8) leads to

$$\Pi = \frac{1}{(1+R_f)^T}\int_0^{\frac{a-q^*}{b}} q^* p f_{RN}(p)\mathrm{d}p + \frac{1}{(1+R_f)^T}\int_{\frac{a-q^*}{b}}^{\frac{a}{b}} q^*\frac{a-q^*}{b} f_{RN}(p)\mathrm{d}p. \tag{29}$$

Letting again the retailer be characterized by CARA risk aversion, $r_u(\cdot) = \alpha$, then Eq. (28) simplifies to

$$q\alpha = \frac{\mathrm{d}}{\mathrm{d}p}\left(\ln\frac{f_P(p)}{f_{RN}(p)}\right) \quad \text{for } 0 \leqslant p < \frac{a-q}{b} \tag{30}$$

and, therefore

$$f_P(p)\mathrm{e}^{-[q\alpha p+\beta]} = f_{RN}(p) \quad \text{for } 0 \leqslant p < \frac{a-q}{b}. \tag{31}$$

Since $\int_0^\infty f_P(p)\mathrm{d}p = 1$, we have also another equation in the integration constant β

$$\int_0^\infty f_{RN}(p)\mathrm{e}^{[q\alpha p+\beta]}\mathrm{d}p = \int_0^\infty f_P(p)\mathrm{d}p = 1 \quad \text{or}$$
$$\mathrm{e}^{-\beta} = \int_0^\infty f_{RN}(p)\mathrm{e}^{q\alpha p}\,\mathrm{d}p = M_{RN}(q\alpha), \tag{32}$$

where $M_{RN}(q\alpha)$ is the moment generating function (MGF) of the risk neutral distribution. To highlight this result, we consider again a numerical example. Explicitly, let again the risk neutral distribution to be normal with mean and standard deviation: $\mu := \frac{a}{2b}$, $\sigma := \frac{a}{10b}$ and use the system parameters: $R = 0.1$, $T = 1$, $\Pi = 10$, $a = 30$, $b = 3$, $\alpha = 0.1$. Then using (32) and solving (29) with Maple, we find the subjective probability distribution (see Fig. 3)

$$f_P(p) = \frac{1}{2}\frac{\sqrt{2}\mathrm{e}^{\left(-\frac{(p-5)^2}{2}\right)}\mathrm{e}^{(0.2200002078p-1.124200119)}}{\sqrt{\pi}} \tag{33}$$

and the optimal order quantity is $q^* = 2.200002078$ implying an order bias of the retailer which is equal to $q_p = 2.107294967$. From Fig. 3 we observe again a lag between the two distributions; the private distribution however is not as much overestimated, as in Fig. 2.

The results obtained in this example, are of course sensitive to the retailer risk attitude which was assumed constant. If instead of a CARA we assume a DARA (Decreasing Absolute Risk Aversion) risk attitude our results will differ. For simplicity assume a logarithmic (DARA) utility function, $\ln(\Omega(q,P(p)))$, with $r_u[\Omega(q,P(p))] = \frac{1}{\Omega(q,P(p))}$. Then Eq. (28) takes the following form:

$$\frac{1}{p} = \frac{\mathrm{d}}{\mathrm{d}p}\left(\ln\frac{f_P(p)}{f_{RN}(p)}\right) \quad \text{for } 0 \leqslant p < \frac{a-q}{b}. \tag{34}$$

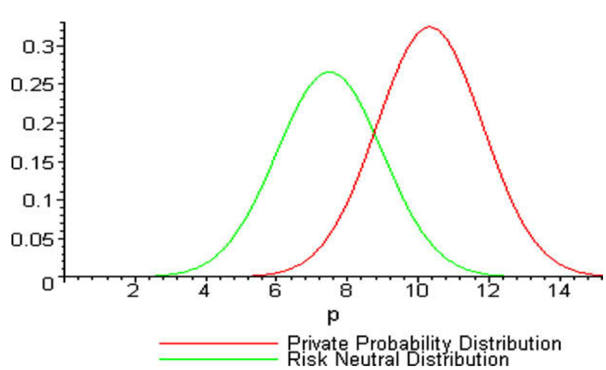

Fig. 2. The private and the risk neutral distributions.

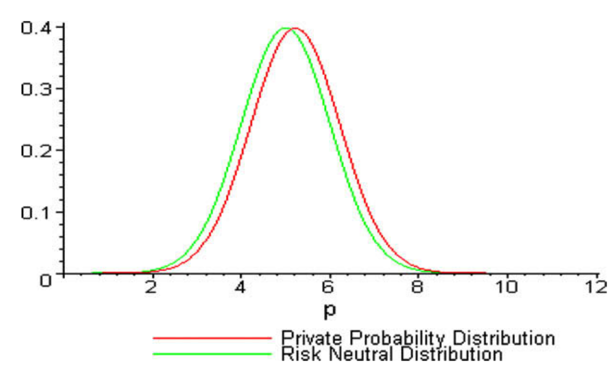

Fig. 3. The probability distributions and price promotion with a CARA attitude.

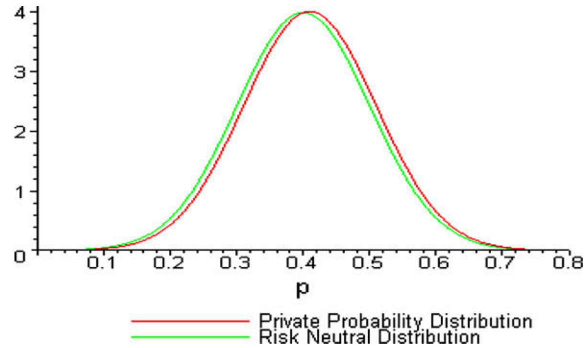

Fig. 4. Probability distributions in case of price promotion and DARA.

Or $\frac{f_P(p)}{f_{RN}(p)} = p + \beta$ for $0 \leqslant p < \frac{a-q}{b}$. Since $\int_0^\infty f_P(p)\mathrm{d}p = 1$, we have as a result, $\int_0^\infty (p+\beta) f_{RN}(p)\mathrm{d}p = \int_0^\infty f_P(p)\mathrm{d}p = 1$, determining the integration constant

$$\beta = 1 - E[\pi]. \tag{35}$$

Thus, unlike a CARA risk attitude, for a DARA risk attitude the price promotional strategy in case of excess inventories results in forecasts to be independent of the optimal order quantity. In this case

$$f_P(p) = (p + 1 - E[\pi]) f_{RN}(p). \tag{36}$$

Fig. 4 illustrates the relationship for normal distribution with parameters: $\mu = 0.4$ and $\sigma = 0.1$.

4. Conclusions and extensions

The increased prominence of financial markets and their widespread use in pricing and in risk allocation provides an opportunity for convergence between financial tools and operational problems, the latter based mostly on a private valuation of costs and benefits associated to a specific policy. This paper has sought to use such an approach in dealing with an order policy and assess its price in terms of market beliefs when markets are complete. Namely when end goods are traded competitively, there is no arbitrage and market process is unique. In this case, the risk neutral framework may be applied provided that the risk neutral distribution can be determined. While such a distribution is difficult to assess, it is also implied in retailers private forecasts of future risk prices and their risk attitudes. We have used such a relationship and proved its specific characteristic in the context of an order policy (and complete markets) to determine one distribution (private) on the other (risk neutral) and vice versa. In addition, a number of examples were employed to provide both a motivation and specific results based on our convergence approach to finance and operations management.

The approach used has however far greater implications that stated in the paper, for an estimate of an implied risk neutral distribution. It provides the means to price as well derivative products such as options of various sorts on the same product that retailers trade. Such an approach was pointed out in the control of inventory as early as 1986 by Ritchken and Tapiero (1986) and has since been used in numerous papers that seek to price optional contracts in operations management. In the Ritchken–Tapiero model, however, the risk neutral distribution was assumed known and provided by the market rather than implied as it is the case in this paper. A direct extension of this paper (see also Kogan and Tapiero, 2007 and Tapiero 2007) would of course be to use the risk neutral distribution to determine an order policy combined with optional contracts (such as Call and Put options to mitigate the effects of excess and inventory shortage). For simplicity such problems were not considered here but are obviously a direct implication of the paper's result.

In addition, operations management (ordering and inventory policies) usually involves more than price and demand uncertainty (with demand a function of price). For example, if $S(\pi) = \xi - b\pi$, then the parameters ξ and b may themselves be random and independent of the random price. Such issues will naturally complicate our results but within the framework stated here are nonetheless tractable. Say for convenience that b is a known parameter while ξ is a random variable dependent of prices with a conditional distribution function, $f(\xi|p)$. In this case, our initial problem (4) and (2), assumes the following form:

$$\max_{P\geqslant 0, q\geqslant 0} J(P,q) = \max_{P\geqslant 0, q\geqslant 0} \int_0^\infty \int_0^\infty U[\Omega(q,P(p),\xi)] f(\xi|p) f_P(p)\mathrm{d}\xi\,\mathrm{d}p$$

$$\text{Subject to} \quad \Pi = \frac{1}{(1+R_f)^{\mathrm{T}}} \int_0^\infty \int_0^\infty \Omega(q,P(p),\xi) f(\xi|p) f_{RN}(p)\mathrm{d}\xi\,\mathrm{d}p.$$

A Variational analysis of this problem, albeit more complicated will reveal that

$$f_P(p) \int_0^\infty U'[\Omega(q,P(p),\xi)] \frac{\partial \Omega(q,P(p),\xi)}{\partial P(p)} f(\xi|p)\mathrm{d}\xi = v \frac{f_{RN}(p)}{(1+R_f)^{\mathrm{T}}} \int_0^\infty \frac{\partial \Omega(q,P(p),\xi)}{\partial P(p)} f(\xi|p)\mathrm{d}\xi$$

where v is a Lagrange multiplier. Differentiating with respect to p, one can eliminate constant v, and thereby obtain a relationship between the retailer's risk aversion, his private forecast of prices and thereby his implied risk neutral distribution. In such a case, analytical results will be more difficult but numerical analyses are still possible.

Appendix

Proof of Proposition 1. We introduce first the new variable $X(p)$ which satisfies the following differential equation:

$$\dot{X}(p) = \frac{1}{(1+R_f)^{\mathrm{T}}} \Omega(q,P(p)) f_{RN}(p), \quad X(0) = 0, \;\; X(\infty) = \Pi.$$

To apply the maximum principle, we construct the Hamiltonian

$$H = U[\Omega(q,P(p))] f_P(p) + \psi(p) \frac{1}{(1+R_f)^{\mathrm{T}}} \Omega(q,P(p)) f_{RN}(p),$$

where the co-state variable $\psi(p)$ satisfies the differential equation

$$\dot{\psi}(p) = -\frac{\partial H}{\partial X} = 0$$

with no boundary conditions, i.e., the co-state variable is a constant, which for convenience is denoted as $\psi(p) \equiv -\lambda$ for any p. This implies that the shadow price for making one more dollar in profit is constant.

If $U(\cdot)$ and $\Omega(\cdot)$ are concave functions in $P(p)$ and there exists an interior solution, then the optimal price $P(p)$ is found by the Maximum Principle by differentiating the Hamiltonian with respect to $P(p)$

$$U'[\Omega(q,P(p))] \frac{\partial \Omega(q,P(p))}{\partial P(p)} f_P(p) - \lambda \frac{1}{(1+R_f)^{\mathrm{T}}} \frac{\partial \Omega(q,P(p))}{\partial P(p)} f_{RN}(p) = 0$$

or

$$U'[\Omega(q,P(p))] f_P(p) - \lambda \frac{1}{(1+R_f)^{\mathrm{T}}} f_{RN}(p) = 0, \quad p \geqslant 0. \tag{A.1}$$

This latter equation implies a relationship between the marginal utility and both risk neutral and the private probability distributions for any possible price realization p. Note, that if the profit $\Omega(q,p)$ does not depend on price for an interval of prices, then retailer's risk attitude does not affect the optimality condition.

Therefore $f_P(p)$ in this interval can be chosen arbitrarily so that the integral over the entire function be equal to one (as illustrated in Section 3).

Since the condition (A.1) must hold for any $p \geqslant 0$, assuming that the probability density functions are differentiable almost everywhere and differentiating (A.1) with respect to p we find

$$U''[\Omega(q,P(p))]\frac{\partial\Omega(q,P(p))}{\partial p} = \frac{\lambda}{(1+R_f)^{\mathrm{T}}} \times \frac{f'_{\mathrm{RN}}(p)f_{\mathrm{P}}(p) - f_{\mathrm{RN}}(p)f'_{\mathrm{P}}(p)}{[f_{\mathrm{P}}(p)]^2}. \tag{A.2}$$

Combining (A.1) and (A.2), and employing the Arrow-Pratt measure (coefficient) of absolute risk aversion, $r_u(\cdot)$ we obtain:

$$r_u[\Omega(q,P(p))] = -\frac{U''}{U'} = \left(\frac{f'_{\mathrm{P}}(p)}{f_{\mathrm{P}}(p)} - \frac{f'_{\mathrm{RN}}(p)}{f_{\mathrm{RN}}(p)}\right)\frac{1}{\frac{\partial\Omega(q,P(p))}{\partial p}}, \tag{A.3}$$

and therefore, we obtain the condition stated in the proposition:

$$\frac{\partial\Omega(q,P(p))}{\partial p} r_u[\Omega(q,P(p))] = \frac{\mathrm{d}}{\mathrm{d}p}(\ln f_{\mathrm{P}}(p) - \ln f_{\mathrm{RN}}(p)) = \frac{\mathrm{d}}{\mathrm{d}p}\left(\ln\frac{f_{\mathrm{P}}(p)}{f_{\mathrm{RN}}(p)}\right), \quad p \geqslant 0. \qquad \square \tag{A.4}$$

References

Ait-Sahalia, Y., Lo, A., 1998. Nonparametric estimation of state-price densities implicit in financial asset prices. Journal of Finance 53, 499–548.

Alonzo F., 2005, Testing the forecasting performance of IBEX 35 option implied risk neutral densities, Working Paper, Bank of Spain.

Jackwerth, J.C., 1999. Option implied risk neutral distributions and implied binomial trees: a literature review. Journal of Derivatives 7, 66–82.

Jackwerth, J.C., 2000. Recovering risk aversion from option prices and realized returns. The Review of Financial Studies 13 (2), 433–451.

Kogan, K., Tapiero, C.S., 2007. Supply Chains, Operations Modeling and Management. Springer-Verlag, Boston.

Munier, B., Tapiero, C.S., 2008. Risk Attitudes, Encyclopedia of Quantitative Risk Assessment and Analysis. Wiley, New York and London.

Oum,Yumi, Shmuel, Oren, Deng, Shijie, 2006. Hedging quantity risks with standard power options in a competitive wholesale electricity market, In: Birge, John, Linetsky, Vadim (Issue Eds.), Special Issue on Applications of Financial Engineering in Operations, Production, Services, Logistics, and Management. Naval Research Logistics, vol. 53(7), pp. 697–712.

Perignon, C., Villa, C., 2002. Extracting information from option markets: Smiles, state price densities and risk aversion, May, European Financial Management.

Ritchken, P., Tapiero, C.S., 1986. Contingent claim contracts and inventory control. Operations Research 34, 864–870.

Tapiero, C.S., 2005. Risk and Financial Management: Mathematical and Computational Methods. Wiley, London and New York.

Tapiero Oren, 2007, Testing techniques and forecasting ability of FX options implied risk neutral densities, Masters Thesis, Department of Economics and Finance, University of Montrel, Montreal, Canada.

Tapiero, C.S., 2007. Market pricing orders management with price and demand uncertainty in complete markets, International Journal of Production Economics, re-submitted for publication.