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Allocating multiple defensive resources in a zero-sum game setting

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Abstract This paper investigates the problem of allocating multiple defensive resources to protect multiple sites against possible attacks by an adversary. The effectiveness of the resources in reducing potential damage to the sites is assumed to vary across the resources and across the sites and their availability is constrained. The problem is formulated as a two-person zero-sum game with piecewise linear utility functions and polyhedral action sets. Linearization of the utility functions is applied in order to reduce the computation of the game's Nash equilibria to the solution of a pair of linear programs (LPs). The reduction facilitates revelation of structure of Nash equilibrium allocations, in particular, of monotonicity properties of these allocations with respect to the amounts of available resources. Finally, allocation problems in non-competitive settings are examined (i.e., situations where the attacker chooses its targets independently of actions taken by the defender) and the structure of solutions in such settings is compared to that of Nash equilibria.

Keywords Multiple resource allocation · Resource substitution · Nash equilibria · Allocation monotonicity

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1 Introduction

The problem of allocating multiple scarce resources to optimize an objective function, in particular a linear one, is one of the classic problems in Operations Research. These problems can be traced back to the foundation of linear programming (LP); for example, early work of Dantzig, Hitchcock, Kantorovich and Koopmans among others (see Cottle et al. 2007). The surveys of Luss (1999) and Katoh and Ibaraki (1998), the recent textbook of Luss (2012) and references therein provide a perspective on the extensive study of this problem.

In most of the work to date on allocating resources a decision maker seeks to maximize a single objective function. In contrast, the current paper considers a *zero-sum* game theoretic variant of the problem where some parameters of an agent's utility function are determined by an adversary who attempts, in turn, to minimize the utility of the first agent. Here, two parties select their actions simultaneously. Each party takes into account the potential effect of the actions of the other party. The motivation for the problem we study is a scenario where a defender *has* to allocate scarce resources of limited capacity to reduce the vulnerability of n sites. The resources are operationally substitutable but their (possibly varying) effectiveness levels are not uniform. An attacker decides whether to attack, and chooses a randomized strategy that assigns probabilities for attacking each site.

In competitive environments, when outcomes depend on actions of multiple interested parties, Nash equilibria are considered as reasonable outcomes, and their determination replaces the computation of "optimal solutions" in non-competitive environments. Resource allocation in adversarial settings has been studied in the context of the military operations research and conflict economics; see for example Blackett (1954), Roberson (2006), and Franke and Öztürk (2009). The current paper addresses an attacker-defender setting similar to that considered by Golany et al. (2009) where the defender has to satisfy hard budgetary constraints. Here, the analysis is extended to a more general setting with multiple resources. Also, additional structural properties of Nash equilibria are revealed. Differences in the solution structure of similar allocation problems in competitive environments vs. non-competitive environments have already been explored for the case of a single resource; see for example Bier et al. (2008), Golany et al. (2009), Powell (2007). Note that the explicit computation of Nash equilibria is generally a challenging task, and even more so when one seeks all Nash equilibria (see for example the commonly cited paper of Papadimitriou 2001). Using linearization and a classical result for polyhedral games (Wolfe 1956) we efficiently reduce the computation of Nash equilibria of our game to the solution of a pair of LPs.

In practice, the attacker and defender may confront each other over multiple time periods. In this paper we model a static game that may also correspond to a stage game in a multistage dynamic game. Note that when played over multiple time periods this game would not fit within the well-studied repeated game setting. This is due to the payoffs and action sets that may change from one stage to another (for example since the budgets and existing defenses may change over time). Nevertheless, a main contribution of this paper has implications for practical multistage applications, as well as for multistage modeling extensions; it includes the investigation of monotonicity properties of the allocated resource amounts in the budget of the defender. Monotonicity is a desirable property implying that the defender will not have to shift existing defenses from one site to another at a later stage in order to be in a Nash equilibrium. Specifically, we show that the Nash-equilibrium allocations increase in amount of a single available resource. On the other hand, with two resources, a simple example illustrates that the allocations are non-monotone in the amounts of available resources. Nevertheless, a surprising weak form of monotonicity is proven in general (with arbitrarily many resources).

In the next section we introduce the notation and model of the game. Then, some properties of optimal solutions of non-competitive resource allocation problems are examined to serve as a benchmark for comparison with the game setting. We continue to explore structure of Nash equilibria, which then lends itself to prove the main result about monotonicity of Nash equilibrium allocations.

2 The model

The set of reals, positive reals and nonnegative reals are denoted $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_\oplus$, respectively. Finally, for $z \in \mathbb{R}$, let $z_+ \equiv \max\{z, 0\}$.

We consider a *resource allocation game* with two players referred to as *defender* and *attacker* and denoted X and W, respectively. The defender X has a limited amounts of m resources indexed by $j \in M = \{1, \dots, m\}$ with the available amount of resource j being $C_j \geq 0$. The defender *may* allocate its resources to the defense of n sites indexed by $i \in N \equiv \{1, \dots, n\}$. The set of actions available to defender is then represented by

$$\mathcal{X} \equiv \left\{ x \in \mathbb{R}_\oplus^{n\times m} \,\middle|\, \sum_{i\in N} x_{ij} \leq C_j \text{ for all } j \in M \right\} \tag{1}$$

(the inequalities constraining the defender's use of resources do not require resource-depletion). The feasible actions for the attacker W are probability vectors from

$$\mathcal{W} \equiv \left\{ w \in \mathbb{R}_\oplus^{n} \,\middle|\, \sum_{i\in N} w_i \leq 1 \right\}; \tag{2}$$

for $w \in W$, w_i the probability of attacking site $i \in N$, and $1 - \sum_{i\in N} w_i$ corresponds to the probability of not attacking at all. It is important to note that the attacker does not observe the defender's decisions *before* making its own decisions, i.e., the attacker is not optimizing its actions with respect to known *defenses*. Similarly, the defender is not optimizing with respect to a given attack or any fixed randomized strategy. Rather, both sides participate in a game where decisions are made simultaneously.

The utility functions of W and X are expressed in terms of a vector $b \in \mathbb{R}_+^m$ and a matrix $a \in \mathbb{R}_\oplus^{m\times n}$. Here, $b_i > 0$ is the expected cost of damage to site $i \in N$ if attacked while no resources are allocated to its protection, and $a_{ij} \geq 0$ is the reduction in the expected cost of damage to site $i \in N$, per unit investment of resource $j \in M$; see Golany et al. (2009) for further details when $m = 1$. The utility function of the attacker and the defender are then given by functions $u_W : \mathcal{X} \times \mathcal{W} \to \mathbb{R}$ and $u_X : \mathcal{X} \times \mathcal{W} \to \mathbb{R}$ with

$$u_W(x, w) = \sum_{i\in N} w_i \left(b_i - \sum_{j\in M} a_{ij} x_{ij} \right)_+ \quad \text{and} \quad u_X(x, w) = -u_W(x, w). \tag{3}$$

Due to the $(\cdot)_+$, these functions are not linear in x, i.e., the defender's actions; here $(\cdot)_+$ is used to reflect the assumption that there is no value in overprotecting a site.

Tables 1 and 2 summarize the data of the problem and the decision variables of the two players.

We say that $(x^*, w^*) \in \mathcal{X} \times \mathcal{W}$ is a Nash equilibrium if x^* is the defender's *best response* to w^* and w^* is the attacker's best response to x^*, i.e.,

$$u_X(x^*, w^*) = \max_{x\in\mathcal{X}} u_X(x, w^*) \quad \text{and} \quad u_W(x^*, w^*) = \max_{w\in\mathcal{W}} u_W(x^*, w). \tag{4}$$

Table 1 The data of the problem

Resource	Damage reduction rates				Resource availability
	Site				
	1	2	…	n	
1	a_{11}	a_{12}	…	a_{1n}	C_1
2	a_{21}	a_{22}	…	a_{2n}	C_2
⋮	⋮	⋮	⋮	⋮	⋮
m	a_{m1}	a_{m2}	…	a_{mn}	C_m
	Initial damage				
	b_1	b_2	…	b_n	

Table 2 Decision variables of the defender X and of the attacker W

Resource	Allocations of X			
	Site			
	1	2	…	n
1	x_{11}	x_{12}	…	x_{1n}
2	x_{21}	x_{22}	…	x_{2n}
⋮	⋮	⋮	⋮	⋮
m	x_{m1}	x_{m2}	…	x_{mn}
	Attack probabilities of W			
	w_1	w_2	…	w_n

Without loss of generality, it will be assumed throughout that

$$b_1 \geq b_2 \geq \cdots \geq b_n > 0; \tag{5}$$

further, to avoid discussing degenerate situations, it will be assumed that the inequalities in (5) are strict. In the next section, we consider multiple resource allocation in a non-competitive setting in order to facilitate a comparison with the game setting.

3 Non-competitive settings

In this section it is assumed that the n sites are to be secured from a probabilistic threat (e.g., of a natural disaster) using m resources. Let $\pi \in \mathbb{R}^n$ be a (fixed) probability vector where π_i is the probability that target i is exposed to the threat. The data of the problem, as in the game setting, includes for each $i \in N$ an expected cost if a site is exposed to the threat $b_i > 0$ and for each resource $j \in M$ an effectiveness $a_{ij} \geq 0$. If $x = (x_{ij})$ is allocated, then the damage to site i if exposed to the threat is $(b_i - \sum_{j \in M} a_{ij} x_{ij})_+$. Given the available resource amounts $C \in \mathbb{R}^m_+$, to be allocated to the defense of the sites, a decision maker faces the problem

$$\hat{\theta}(\pi) = \min_{x \in \mathcal{X}} \left\{ \sum_{i \in N} \pi_i \left(b_i - \sum_{j \in M} a_{ij} x_{ij} \right)_+ \right\}, \tag{6}$$

where $\mathcal{X}$ is defined by (1). We now consider the linearized optimization problem:

$$\theta'(\pi) = \min \sum_{i\in N} \pi_i \left(b_i - \sum_{j\in M} a_{ij} x_{ij} \right) \tag{7a}$$

$$\text{subject to} \quad \sum_{i\in N} x_{ij} \le C_j \qquad j \in M \tag{7b}$$

$$\sum_{j\in M} a_{ij} x_{ij} \le b_i \qquad i \in N \tag{7c}$$

$$x \ge 0. \tag{7d}$$

The next result relates optimal solutions of (6) and optimal solutions of (7a)–(7d). Its proof is given in Appendix B.

Proposition 1

(a) $\hat{\theta}(\pi) = \theta'(\pi)$.
(b) *A feasible solution of* (7a)–(7d) *is optimal for* (7a)–(7d) *if and only if it is optimal for* (6).

The next proposition bounds the number of targets that are partially protected in optimal solutions of (7a)–(7d). The proof is given in Appendix C. For $x \in \mathcal{X}$ let $L(x) \equiv \{i \in N \mid \sum_{j\in M} a_{ij} x_{ij} = 0\}$, $U(x) \equiv \{i \in N \mid \sum_{j\in M} a_{ij} x_{ij} = b_i\}$ and $\nu(x) \equiv \{i \in N \mid 0 < \sum_{j\in M} a_{ij} x_{ij} < b_i\}$.

Proposition 2 *There exists an optimal solution x' of* (7a)–(7d) *such that $\nu(x') \le m$.*

The result of Proposition 2 is used for comparison with the structure of Nash equilibria that is studied in the next section.

4 Characterization, computation and structure of Nash equilibria (through linearization)

This section shows how Nash equilibria of the original game, defined by (1)–(3), can be computed through the use of Linear Programming (LP). For this purpose we consider the game where the utility functions are modified by dropping the symbol $(\cdot)_+$ in (3); we denote the new utility function $\hat{u}_X$ and $\hat{u}_W$, respectively, and refer to them as the *linearized utility functions*. Specifically, for $(x, w) \in \mathcal{X} \times \mathcal{W}$,

$$\hat{u}_W(x, w) = \sum_{i\in N} w_i \left(b_i - \sum_{j\in M} a_{ij} x_{ij} \right) \quad \text{and} \quad \hat{u}_X(x, w) = -\sum_{i\in N} w_i \left(b_i - \sum_{j\in M} a_{ij} x_{ij} \right); \tag{8}$$

We refer to the *the linearized game* as the game with the new utility functions. Elementary facts about Nash equilibria of the original and linearized game are summarized in the next proposition.

The following proposition records standard results that follow from $u_W(x, w)$ and $\hat{u}_W(x, w)$ being concave and continuous in x and linear in w and $\mathcal{X}$ and $\mathcal{W}$ being compact and convex, see Berkovitz (2002, p. 117).

Proposition 3

(a) $V \equiv \min_{x\in\mathcal{X}} \max_{w\in\mathcal{W}} u_W(x,w) = \max_{w\in\mathcal{W}} \min_{x\in\mathcal{X}} u_W(x,w)$.
(b) (x^*, w^*) *is a Nash equilibrium of the original game if and only if*

$$x^* \in \operatorname*{argmin}_{x\in\mathcal{X}}\Big[\max_{w\in\mathcal{W}} u_W(x,w)\Big] \quad \textit{and} \quad w^* \in \operatorname*{argmax}_{w\in\mathcal{W}}\Big[\min_{x\in\mathcal{X}} u_W(x,w)\Big]. \tag{9}$$

(c) *The original game has a Nash equilibrium.*
(a′) $\hat{V} \equiv \min_{x\in\mathcal{X}}\ \max_{w\in\mathcal{W}} \hat{u}_W(x,w) = \max_{w\in\mathcal{W}}\ \min_{x\in\mathcal{X}} \hat{u}_W(x,w)$.
(b′) (x^*, w^*) *is a Nash equilibrium of the linearized game if and only if*

$$x^* \in \operatorname*{argmin}_{x\in\mathcal{X}}\Big[\max_{w\in\mathcal{W}} \hat{u}_W(x,w)\Big] \quad \textit{and} \quad w^* \in \operatorname*{argmax}_{w\in\mathcal{W}}\Big[\min_{x\in\mathcal{X}} \hat{u}_W(x,w)\Big]. \tag{10}$$

(c′) *The linearized game has a Nash equilibrium.*

Part (b) of Proposition 3 demonstrates that (x^*, w^*) is a Nash equilibrium of the original game if and only if x^* and w^* satisfy, respectively, the two independent conditions of (9). Hence, we refer to *equilibrium allocations* of the attacker and of the defender as independent properties characterized by the corresponding condition in (9). Part (b′) establishes the same conclusions for the linearized game.

Henceforth, let V and $\hat{V}$ be the *values* of the *original* and *linearized game*, respectively, as defined in Proposition 3. The next result links Nash equilibria of the original and the linearized games. The proof is given in Appendix D.

Proposition 4

(a) *Every Nash equilibrium* (x^*, w^*) *of the linearized game, is a Nash equilibrium of the original game.*
(b) $V = \hat{V} \geq 0$.
(c) *If* $V > 0$, *then the set of Nash equilibria of the original game and the linearized game coincide.*

The next example demonstrates that the condition $V > 0$ cannot be dropped from Proposition 4(b).

Example 1 Consider a single site and a single resource, with $b_1 = a_{11} = 1$ and $C_1 = 2$. Then $V = \hat{V} = 0$ and $(x^*_{11} = 1, w^*_1 = 1)$ is a Nash equilibrium for the original game. But, it is not a Nash equilibrium for the linearized game as $\hat{u}_W(2,1) = -1 < \hat{u}_W(1,1)$. Still, the (unique) Nash equilibrium $(1,0)$ of the linearized game is a Nash equilibrium of the original game.

The next result characterizes Nash equilibria of the linearized game and shows that its equilibria can be determined by solving two LPs. The LPs of interest are

$$\theta^* = \min \theta \tag{11a}$$

$$\text{s.t.} \quad \theta + \sum_{j\in M} a_{ij}x_{ij} \geq b_i, \quad i \in N, \tag{11b}$$

$$\sum_{i\in N} x_{ij} \leq C_j, \quad j \in M, \tag{11c}$$

$$\theta, x_{ij} \geq 0, \quad i \in N, j \in M \tag{11d}$$

and its dual:

$$\max \sum_{i\in N} b_i w_i - \sum_{j\in M} C_j \xi_j \tag{12a}$$

$$\text{s.t.} \quad \sum_{i\in N} w_i \le 1 \tag{12b}$$

$$a_{ij} w_i \le \xi_j, \quad i \in N, j \in M, \tag{12c}$$

$$w_i, \xi_j \ge 0, \quad i \in N, j \in M. \tag{12d}$$

We note that $(x, \theta) = (0, b_1)$ is feasible for the LP (11a)–(11d) whose objective function is bounded below by 0. Hence, this LP and its dual (12a)–(12d) always have an optimal solution. The proof of the next proposition is given in Appendix E.

Proposition 5

(a) (x^*, w^*) *is a Nash equilibrium of the linearized game if and only if*

$$x^* \in \operatorname*{argmin}_{x\in\mathcal{X}} \left\{ \max_{w\in\mathcal{W}} \left\{ \sum_{i=1}^{n} w_i \left(b_i - \sum_{j\in M} a_{ij} x_{ij} \right) \right\} \right\}$$
$$= \operatorname*{argmin}_{x\in\mathcal{X}} \left\{ \max_{i\in N} \left\{ \left(b_i - \sum_{j\in M} a_{ij} x_{ij} \right)_+ \right\} \right\} \tag{13}$$

and

$$w^* \in \operatorname*{argmax}_{w\in\mathcal{W}} \left\{ \min_{x\in\mathcal{X}} \left\{ \sum_{i\in N} w_i \left(b_i - \sum_{j\in M} a_{ij} x_{ij} \right) \right\} \right\}. \tag{14}$$

(b) x^* *satisfies* (13) *if and only if* (x^*, θ^*) *is optimal for* (11a)–(11d) *for some* $\theta^* \in \mathbb{R}$, w^* *satisfies* (14) *if and only if* (w^*, ξ^*) *is optimal for* (12a)–(12d) *for some* $\xi^* \in \mathbb{R}^m$, *and the common optimal objective values of LPs* (11a)–(11d) *and* (12a)–(12d) *equals* $\hat{V}$.

(c) *There exists* x^* *in* $\mathcal{X}$ *that satisfies* (13) *and* $b_i - \sum_{j\in M} a_{ij} x^*_{ij} \ge 0$ *for all* $i \in N$.

Proposition 5 demonstrates that finding Nash equilibria for the linearized game reduces to solving LPs (11a)–(11d) and (12a)–(12d). By Proposition 4(a), resulting solutions are Nash equilibria of the original game. Proposition 4(b) further demonstrates that if the optimal objective values of the LP's are positive, Nash equilibria of the original game are given by optimal solutions of the LPs (11a)–(11d) and (12a)–(12d).

Suppose (x^*, θ^*) is optimal for (11a)–(11d). Then $\hat{V} = \theta^*$. Further, if $\theta^* > 0$, then x^* is an equilibrium allocation for X under both the linearized and the original models and solutions of LP (12a)–(12d) yield equilibrium allocations for W. Alternatively, if $\theta^* = 0$, then $(w^*, \xi^*) = (0, 0)$ solves LP (12a)–(12d) and $w^* = 0$ is an equilibrium allocation for W under the linearized model. In this case, there are solutions x to (11b)–(11d) with $\theta = 0$ and each such x is an equilibrium allocation for X in the linearized game and therefore in the original game; in particular, there are such solutions x where the inequalities in (11b) hold as equalities, representing allocations where no site is over-protected. As for W, when $\theta^* = 0$, any $w^* \in \mathcal{W}$ is an equilibrium allocation for W under the original model.

For $x \in R_{\oplus}^{n\times m}$, let

$$I(x) \equiv \operatorname*{argmax}_{i\in N} \left\{ b_i - \sum_{j\in M} a_{ij} x_{ij} \right\} \quad \text{and} \quad P(x) \equiv \left\{ i \in N \,\middle|\, \sum_{j\in M} x_{ij} > 0 \right\}. \tag{15}$$

A set $S \subseteq N$ is said to be *consecutive* if it is one of the n sets: $[1], [2], \ldots, N$. The next lemma identifies structure present in optimal solutions of (11a)–(11d). Its proof is included in Appendix F.

Lemma 1

(a) *For some optimal solution* (x^*, θ^*) *of* (11a)–(11d): $I(x^*) = P(x^*) \cup \{i \in N \mid b_i = \theta^*\}$ *and the sets* $I(x^*)$ *and* $P(x^*)$ *are consecutive.*

Further, assuming $\theta^* > 0$:

(b) *For every optimal solution* (x^*, θ^*) *of* (11a)–(11d): $\theta^* = \max_{i\in N}\{b_i - \sum_{j\in M} a_{ij} x^*_{ij}\}$ *and* $I(x^*) = \{i \in N : b_i - \sum_{j\in M} a_{ij} x^*_{ij} = \theta^*\} \subseteq P(x^*) \cup \{i \in N \mid b_i = \theta^*\}$.
(c) *If* $a > 0$, *then for every optimal solution* (x^*, θ^*) *of* (11a)–(11d): $I(x^*) = P(x^*) \cup \{i \in N \mid b_i = \theta^*\}$, $I(x^*)$ *and* $P(x^*)$ *are consecutive and* $\sum_{i\in N} x^*_{ij} = C_j$ *for each* $j \in M$.
(d) *For some optimal solution* (x^*, θ^*) *of* (11a)–(11d): $\sum_{i\in P(x^*)}[|\{j \in M \mid x^*_{ij} > 0\}| - 1] \le m - 1$, *in particular, there are at most* $m - 1$ *targets* i *with* $|\{j \in M \mid x^*_{ij} > 0\}| > 1$.

Lemma 1 implies the following structure of a Nash equilibrium for our model: with $a > 0$ and an optimal objective value of (11a)–(11d) $\theta^* > 0$, the protection level of each target i is either θ^* or $b_i \le \theta^*$ with no investment of any resource in the second category of targets; further, the overlap between the two categories of targets consists of at most a single target. Also, at most $m - 1$ of the targets that are allocated some resource get their allocation from more than a single resource. The proof of part (d) of Lemma 1 (in Appendix F) shows how an equilibrium allocation x^* for the defender that satisfies the conclusion of this part can be computed.

Further, Lemma 1 shows that for the Nash equilibrium x^*, the protection level of each site i is either b_i or a common value θ^* (the optimal objective value of (11a)–(11d)). In contrast to the non-competitive case discussed in Section 3, where an optimal solution x' of (7a)–(7d) *has* at most *m partially protected* sites. This observation is consistent with conclusions of Golany et al. (2009) and Canbolat et al. (2012) that in competitive settings one should distribute the use of resources among many targets whereas in non-competitive settings the optimal allocation tends to focus the effort on smaller subsets of sites.

5 Allocation-monotonicity for the defender

In this section we study Nash equilibria of the (allocation) game, defined by (1)–(3), under variations of the resource vector $C \equiv (C_1, \dots, C_m)^T \in \mathbb{R}^m_+$. To do so, we parameterize the characteristics of the problem by the budget vector C; for example, the set $\mathcal{X}$ of available actions for player X will be indexed by C, and referred to as $\mathcal{X}(C)$.

We say that the game *exhibits equilibrium-monotonicity for* $\mathcal{C} \subseteq \mathbb{R}^m_+$ if for every $C, \bar{C} \in \mathcal{C}$ with $C \le \bar{C}$ and every equilibrium allocation x^* for the defender in the games with resource vector C the defender has an equilibrium allocation $\bar{x}$ in the games with resource vector $\bar{C}$ such that $x^* \le \bar{x}$. We say that the game *exhibits strict equilibrium-monotonicity for* $\mathcal{C}$ if the above assertion holds for every $\bar{x}$ (rather then some $\bar{x}$). The reference to $\mathcal{C}$ will be dropped when $\mathcal{C} = \mathbb{R}^m_+$.

Equilibrium-monotonicity is a desirable property as it facilitates planning in stages when the availability of resources increases over time. In such cases, equilibrium-monotonicity assures that equilibrium allocations determined when an initial budget is given, remain efficient when more of the resources become available, i.e., budget increases do not result in a need to reduce allocations made under an initial budget. The next result considers problems with a single resource, i.e., $m = 1$. In this case, we assume that all a_i's are positive (for otherwise allocations to target i is irrelevant) and we drop the index 1 that corresponds to the single resource.

Proposition 6 *Let $m = 1$ and $a > 0$. Then the resource allocation game exhibits equilibrium-monotonicity and it exhibits strict equilibrium-monotonicity for $\mathcal{C} = [0, \sum_{i\in N} \frac{b_i}{a_i}]$.*

Proof It follows from Luss (1992) or Golany et al. (2009) that for $C \in \mathcal{C}$, a solution to (13) is uniquely determined as the vector x^* with $x_i^* = (\frac{b_i-\theta}{a_i})_+$ for each $i \in N$, where θ is uniquely determined by $C = \sum_{i\in N} \frac{(b_i-\theta)_+}{a_i}$. Clearly, θ strictly increases in $C \le \sum_{i\in N} \frac{b_i}{a_i}$, implying that the x_i^* increases in $C \in \mathcal{C}$, establishing strict equilibrium-monotonicity for $C \in \mathcal{C}$. Next, for $C > \sum_{i\in N} \frac{b_i}{a_i}$, the equilibrium allocations are all the vectors x^* satisfying $\sum_{i\in N} x_i^* \le C$ and $x_i^* \ge \frac{b_i}{a_i}$; in particular, if x^* is such a vector and $\bar{C} \ge C$, then x^* continues to be an equilibrium allocation under $\bar{C}$. Also, if $C \in \mathcal{C}$, $\bar{C} \notin \mathcal{C}$, $C \le \bar{C}$ and x^* is an equilibrium allocation under C, then $x_i^* \le \frac{b_i}{a_i}$ for each $i \in N$ and $\sum_{i\in N} x_i^* \le C$, then the allocation $\bar{x}$ with $\bar{x}_i = \frac{b_i}{a_i}$ for each $i \in N$ is an equilibrium allocation under $\bar{C}$ and satisfies $\bar{x} \ge x^*$. □

The following example illustrates lack of equilibrium-monotonicity when $m \ge 2$.

Example 2 Suppose $N = M = \{1, 2\}$, and

$$a = \begin{pmatrix} 1 & 0.8 \\ 0.6 & 0.2 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \bar{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The only strategy $x^* \in \mathcal{X}(C)$ of player X that satisfies (13) has $x_{11}^* = C_1$ and all other x_{ij}^*'s equal zero; this allocation satisfies $b_1 - \sum_{j=1,2} a_{1j}x_{1j}^* = 2 - 1 \times 1 = 1 = b_2 - \sum_{j=1,2} a_{2j}x_{2j}^*$ and $\max_{i=1,2}\{b_i - \sum_{j=1,2} a_{ij}x_{ij}^*\} = 1$. By Proposition 5, x^* is the only equilibrium allocation of player X. When the available resources are given by $\bar{C} \ge C$, for every allocation $x' \in \mathcal{X}(\bar{C})$ with $x'_{11} \ge x_{11}^* = 1$, we have that $x'_{11} = 1$, $x'_{21} = 0$, $b_1 - \sum_{j=1,2} a_{1j}x'_{1j} = 1 - 0.8x'_{12}$, and $b_2 - \sum_{j=1,2} a_{2j}x'_{2j} = 1 - 0.2x'_{22}$; then

$$\min_{\substack{x\in\mathcal{X}(\bar{C}) \\ x\ge x^*}} \max_{i=1,2} \left\{ b_i - \sum_{j=1,2} a_{ij}x_{ij} \right\} = 0.68,$$

is uniquely attained by $x' = \begin{pmatrix} 1.0 & 0.4 \\ 0.0 & 1.6 \end{pmatrix}$. But, for $x'' \equiv \begin{pmatrix} 0.0 & 2.0 \\ 1.0 & 0.0 \end{pmatrix}$, $\theta'' \equiv \max_{i=1,2}\{b_i - \sum_{j=1,2} a_{ij}x''_{ij}\} = 0.4 < 0.68$; in fact, x'' is the unique allocation satisfying (13). Thus, by Proposition 5, there is no Nash equilibrium $(\hat{x}, \hat{w})$ with $\hat{x} \ge x^*$.

The next result exhibits a weak form of monotonicity when $m > 1$ and the availability of one resource increases without change in the availability of the other resources. We address solutions of (11a)–(11d)—the translation to equilibrium allocations can then be deduced from Propositions 5 and 4. In particular, we shall refer to (11a)–(11d) with a vector $\bar{C} \in \mathbb{R}^m_\oplus$ replacing C as (11) $(\bar{C})$.

Theorem 1 *Assume that $C, \bar{C} \in \mathbb{R}^m_\oplus$, and for some $q \in M$, $\bar{C}_q \ge C_q$ while $\bar{C}_j = C_j$ for all $j \ne q$. Then for every optimal solution (x^*, θ^*) for (11a)–(11d), there exists an optimal solution $(\bar{x}, \bar{\theta})$ for (11) $(\bar{C})$ such that $\bar{x}_{iq} \ge x_{iq}^*$ for all $i \in N$.*

Proof We first consider the case where $a > 0$. Let $\bar{\theta}$ be the optimal objective value of (11) $(\bar{C})$. As every feasible solution of (11a)–(11d) is feasible for (11) $(\bar{C})$, $\theta^* \ge \bar{\theta}$. Further, if $\theta^* = \bar{\theta}$, then (x^*, θ^*) is optimal for (11) $(\bar{C})$ and the conclusion of our proposition holds with $(\bar{x}, \bar{\theta}) = (x^*, \theta^*)$. Henceforth assume that $\bar{\theta} < \theta^*$ which implies that $\theta^* > 0$ and (by part (c) of Lemma 1) for every vector x for which (x, θ^*) is optimal for (11a)–(11d)

$$\sum_{i \in N} x_{ij} = C_j \quad \text{for all } j \in M. \tag{16}$$

We proceed by assuming that $\bar{\theta} > 0$. In this case (again, by part (c) of Lemma 1), for every vector x for which $(x, \bar{\theta})$ is optimal for (11) $(\bar{C})$

$$\sum_{i \in N} x_{ij} = \bar{C}_j \quad \text{for all } j \in M. \tag{17}$$

We next make two observations:

(i) As $P(x^*) \subseteq I(x^*) = \{i \in N : b_i - \theta^* = \sum_{j \in M} a_{ij} x^*_{ij}\}$ (by part (c) of Lemma 1), as $\bar{\theta} < \theta^*$ and as $(\bar{x}, \bar{\theta})$ is feasible for (11b), we have that for each $i \in P(x^*)$, $\sum_{j \in M} a_{ij} x^*_{ij} = b_i - \theta^* < b_i - \bar{\theta} \leq \sum_{j \in M} a_{ij} \bar{x}_{ij}$. Consequently, for each $i \in P(x^*)$, $\bar{x}_{ij'} > x^*_{ij'}$ for some $j' \in M$.

(ii) If $\bar{x}_{ij} > x^*_{ij}$ for $(i, j) \in N \times (M \setminus \{q\})$, then (16)–(17) assure that $\sum_{i \in N} \bar{x}_{ij} = \bar{C}_j = C_j = \sum_{i \in N} x^*_{ij}$; consequently, $\bar{x}_{i'j} < x^*_{i'j}$ for some $i' \in N$.

Let $\bar{x}$ be a minimizer of $|\{(i, j) \in N \times M : x_{ij} \neq x^*_{ij}\}|$ over the vectors $x \in \mathbb{R}^{n \times m}$ that are optimal for (11) $(\bar{C})$. We will show, by contradiction, that $\bar{x}$ satisfies the conclusion of our proposition. So, suppose that $\bar{x}_{uq} < x^*_{uq}$ for some $u \in N$. Then $x^*_{uq} > 0$, assuring that $u \in P(x^*)$ and (by observation (i)), $\bar{x}_{uj'} > x^*_{uj'}$ for some $j' \in M$; as $\bar{x}_{uq} < x^*_{uq}$, $j' \neq q$. Set $i_1 = u$, $j_1 = q$ and $j_2 = j'$. Recursive use of the above two observations allows one to construct sequences $i_1 = u, i_2, i_3, \ldots \in N$ and $j_1 = q, j_2 = j', j_3, \ldots \in M$ with

$$0 \leq \bar{x}_{i_t j_t} < x^*_{i_t j_t} \quad \text{and} \quad \bar{x}_{i_t j_{t+1}} > x^*_{i_t j_{t+1}} \geq 0 \quad \text{for } t = 1, 2, \ldots.$$

Let $\ell \equiv \max\{t = 1, 2, \ldots \mid j_t \notin \{j_1, \ldots, j_{t-1}\}\}$ (finiteness of $m = |M|$ and $j_2 \neq q$ assure $2 \leq \ell \leq m$) and let $p < \ell + 1$ satisfy $j_p = j_{\ell+1}$.

Let $z \in \mathbb{R}^{n \times m}$ be defined by

$$z_{ij} = \begin{cases} \prod_{r=p}^{t-1} \frac{a_{i_r j_r}}{a_{i_r j_{r+1}}} & i = i_t \text{ and } j = j_t \text{ for } t \in \{p, \ldots, \ell\}, \\ -\prod_{r=p}^{t-1} \frac{a_{i_r j_r}}{a_{i_r j_{r+1}}} & i = i_t \text{ and } j = j_{t+1} \text{ for } t \in \{p, \ldots, \ell\}, \\ 0 & \text{otherwise,} \end{cases}$$

where the product over the empty set is defined to be 1. This definition assures that $\sum_{j \in M} a_{ij} z_{ij} = 0$ for each $i \in N$ and with $\alpha \equiv \prod_{r=p}^{\ell} \frac{a_{i_r j_r}}{a_{i_r j_{r+1}}}$,

$$\sum_{i \in N} z_{ij} = \begin{cases} 0 & \text{for } j \in M \setminus \{j_p\}, \\ z_{i_p j_p} + z_{i_\ell j_p} = 1 - \alpha & \text{for } j = j_p. \end{cases}$$

We consider three cases:

Case I: $\alpha < 1$. For $0 < \epsilon \leq \min\{\frac{x^*_{i_t j_t}}{z_{i_t j_t}} : t = p, \ldots, \ell\}$, $(x^* - \epsilon z, \theta^*)$ is feasible for (11a)–(11d) with

$$\sum_{i \in N} (x^* - \epsilon z)_{ij_p} = \sum_{i \in N} x^*_{ij_p} - \epsilon \sum_{i \in N} z_{ij_p} = C_{j_p} - \epsilon(1 - \alpha) < C_{j_p}. \tag{18}$$

For such ϵ, $(x^* - \epsilon z, \theta^*)$ is optimal for (11a)–(11d) while $x = x^* - \epsilon z$ violates (16) for $j = j_p$, a contradiction.

Case II: $\alpha > 1$. For $0 < \epsilon \le \bar{\epsilon} \equiv \min\{\frac{\bar{x}_{i_t j_{t+1}}}{|z_{i_t j_{t+1}}|} : t = p, \dots, \ell\}$ (note that $\bar{\epsilon}$ is well defined as $a > 0$ implies that the denominator is nonzero for all elements in this set), $(\bar{x} + \epsilon z, \bar{\theta})$ is feasible for (11) $(\bar{C})$ with

$$\sum_{i \in N} (\bar{x} + \epsilon z)_{ij_p} = \sum_{i \in N} x^*_{ij_p} + \epsilon \sum_{i \in N} z_{ij_p} = \bar{C}_{j_p} + \epsilon(1 - \alpha) < \bar{C}_{j_p}. \tag{19}$$

For such ϵ, $(\bar{x} + \epsilon z, \bar{\theta})$ is optimal for (11) $(\bar{C})$ while $x = \bar{x} + \epsilon z$ violates (17) for $j = j_p$, a contradiction.

Case III: $\alpha = 1$. As in case II, for $0 < \epsilon \le \bar{\epsilon}$(19) holds with its inequality replaced by equality and $(\bar{x} + \epsilon z, \bar{\theta})$ is feasible and optimal for (11) $(\bar{C})$. From the observation that

$$\begin{aligned} \hat{\epsilon} &\equiv \min\left\{ \left| \frac{\bar{x}_{ij} - x^*_{ij}}{z_{ij}} \right| : (i, j) \in \big\{(i_t, j_t), (i_t, j_{t+1}) \mid t = p, \dots, \ell\big\} \right\} \\ &\le \min\left\{ \frac{\bar{x}_{i_t j_{t+1}}}{|z_{i_t j_{t+1}}|} : t = p, \dots, \ell \right\} = \bar{\epsilon}, \end{aligned}$$

it follows that $(\bar{x} + \hat{\epsilon} z, \bar{\theta})$ is feasible and optimal for (11) $(\bar{C})$. Further, our construction assures that for each $(i, j) \in N \times M$ with $z_{ij} \ne 0$, $(\bar{x} - x^*)_{ij} z_{ij} < 0$ and therefore

$$\operatorname{sgn}\big(\bar{x}_{ij} - x^*_{ij} + \hat{\epsilon} z_{ij}\big) = \begin{cases} \operatorname{sgn}(\bar{x}_{ij} - x^*_{ij}) & \text{if } |\frac{\bar{x}_{ij} - x^*_{ij}}{z_{ij}}| > \hat{\epsilon}, \\ 0 & \text{if } |\frac{\bar{x}_{ij} - x^*_{ij}}{z_{ij}}| = \hat{\epsilon}. \end{cases}$$

Consequently,

$$\Big|\big\{(i, j) \in N \times M : (\bar{x} + \hat{\epsilon} z)_{ij} \ne x^*_{ij}\big\}\Big| < \Big|\big\{(i, j) \in N \times M : \bar{x}_{ij} \ne x^*_{ij}\big\}\Big|,$$

contradicting the selection of $\bar{x}$ and thereby proving that $\bar{x}$ satisfies the conclusion of our proposition.

We next consider the case where $\bar{\theta} = 0 < \theta^*$. We parameterize our analysis by denoting the available amount of resource q by $\beta \in [C_q, \bar{C}_q]$; in particular, the suffix (β) will be added, at convenience, to the characteristics of the problem, e.g., we write $C(\beta) \equiv (C_1, \dots, C_{q-1}, \beta, C_{q+1}, \dots, C_n)$, $\mathcal{X}(\beta) \equiv \mathcal{X}(C(\beta))$, $x^*(\beta) \in \mathcal{X}(C(\beta))$, $\theta^*(\beta) \in \mathbb{R}$, (11) (β) etc. As the coefficient-matrix of (11a)–(11d) has full row rank (after surplus/slack variables are added), the optimal objective value $\theta^*(\beta)$ is piecewise linear and continuous in β with an invariant optimal basis for β in corresponding intervals; further, $\theta^*(\beta)$ is weakly decreasing in β. Let $\hat{\beta} \equiv \sup\{\beta \in [C_q, \bar{C}_q] \mid \theta^*(\beta) > 0\}$. Then, continuity of $\theta^*(\cdot)$ (*see for example* Martin 1975) and $\theta^*(\bar{C}_q) = 0$ assure that $C_q \le \hat{\beta} \le \bar{C}_q$ and $\theta^*(\hat{\beta}) = 0$. For $k = 1, 2, \dots$, let $\beta_k \equiv \hat{\beta} - \frac{\hat{\beta} - C_q}{k}$; in particular, $x^*_{i'1} = C_q$, β_k is increasing in k and $\lim_{k\to\infty} \beta_k = \hat{\beta}$. It follows from the above that for $k \ge 2$ there exists optimal solutions $(x^*(\beta_k), \theta^*(\beta_k))$ of (11) (β_k) such that $x^*(\beta_k)_{iq}$ is weakly increasing in k. Let $\bar{x}$ be any limit point of the sequence $x^*(x^*_{i'1}), x^*(x^*_{i'2}), \dots$ (all $x^*(\beta_k)$'s are in the compact set $\mathcal{X}(\beta)$, hence a limit point exists). Continuity arguments assure that $\lim_{k\to\infty} \theta^*(\beta_k) = \theta^*(\hat{\beta}) = 0 = \bar{\theta}$, and $(\bar{x}, \theta^*(\hat{\beta}) = 0)$ is optimal for (11) $(\hat{\beta})$. Further, from the above result for the case of $\theta^* > 0$ it follows that $\bar{x}_{iq} \ge x^*(x^*_{i'1})_{iq} = x^*_{iq}$ for each $i \in N$. As $\theta^*(\hat{\beta}) = 0 = \theta^*(\bar{C})$ and $C(\hat{\beta}) \le \bar{C}$, it follows that $(\bar{x}, \bar{\theta} = 0)$ is optimal for (11) $(\bar{C})$. So $\bar{x}$ satisfies the conclusion of our proposition.

We finally consider the general case where $a \ge 0$ (rather than $a > 0$). As in the previous paragraph, we use continuity and compactness arguments. For $k \ge 1, 2, \dots$, let $a(k)_{ij} = a_{ij} + \frac{1}{k}$. Let the problem variables and characteristics now be parameterized by the integer k, for example let $(\bar{x}(k), \bar{\theta}(k))$ be an optimal solution of (11) $(\bar{C})$ and $a(k) = (a(k)_{ij})$ replacing

a. It follows from the earlier parts of the proof for $a > 0$ that for $k = 1, 2, \ldots$, there exist $(\bar{x}(k), \bar{\theta}(k))$ such that $\bar{x}(k)_{iq} \geq x^*_{iq}$ for each $i \in N$. Evidently, as k increases, feasibility of (11a)–(11d) becomes tighter and $\bar{\theta}(k)$ increases in k. Let $\bar{\theta}$ be the optimal objective value of (11) $(\bar{C})$. As the set $\mathcal{X}(\bar{C})$ is compact, $\bar{x}(k) \in \mathcal{X}(\bar{C})$ and $\bar{\theta}(k) \in [\bar{\theta}(1), \bar{\theta}]$, for each $k \geq 1$, the sequence $(\bar{x}(k), \bar{\theta}(k))$, for $k = 1, 2, \ldots$, has a limit point, say $(\bar{x}, \bar{\theta})$, continuity arguments then show that $(\bar{x}, \bar{\theta})$ is optimal for (11) $(\bar{C})$ and $\bar{x}_{iq} \geq x^*_{iq}$ for each $i \in N$. □

6 Concluding remarks and future research

In this paper we efficiently reduce (making use of a classical result) the computation of Nash equilibrium of a static defensive resource allocation game with multiple resources, polyhedral action sets, and particular piecewise linear utility functions to the solution of a pair of LPs. This reduction also allows the revelation of structure of Nash equilibria. Monotonicity properties that we establish may have implications for policy making if the game is played over multiple time periods. A lack of monotonicity of two or more resources allocated under a Nash equilibrium when budgets are increased is a negative result with consequences for decision making. On the other hand, the established weaker form of monotonicity states that the allocated amounts of resource that becomes more abundant need not decrease for the defender to be in a Nash equilibrium; it suggests that if there is a choice, one may be better off by increasing the budget of a resource that is more costly to move from one site to another. Another implication is that policy makers may need to set aside sufficient funds for the allocation of resources that are expensive to move at later stages in order to match the increases in budgets of other resources. It is the subject of future research to formally model these considerations in a multistage dynamic game.

The monotonicity issues discussed in this paper address only possible changes in the defender's budget. In future work, we intend to explore situations in which the attacker has multiple units of the attack resource (e.g., in cases where the attacking party is composed of several teams and each team can have its own mixed strategy of attacking multiple sites); accordingly, $\mathcal{W} = \{w \in \mathbb{R}^n : 0 \leq w_i \leq 1 \text{ for each } i \in N, \sum_{i \in N} w_i = W > 1\}$. In this context, one can consider monotonicity with respect to the amount W of attacker's resource.

While the current paper focuses on zero-sum scenarios in which any gain for one is a loss to the other, in future work we intend to extend our work to non zero-sum situations in which each player uses a different set of parameters to evaluate the outcome of the game (similar to Powell 2007 but with multiple rather than a single resource). The insights we hope to gain from exploring non zero-sum games involving the allocation of multiple defensive resources should enable us to treat a case where some of the resources serve only for deception purposes (cf. Levitin and Hausken 2009 which models and analyzes the use of false targets).

In the current paper, the expected damage resulting from a target being attacked decreases linearly in the amount of allocated defensive resource up to a point where a target is fully protected, at that point, the expected damage is constant. The cost of an attack is a specific convex piecewise linear function of the invested resources. In future research, we intend to consider more general convex piecewise linear functions. The modeling and the analysis when such functions are used yields several advantages. First, it may result in closed form solutions for the attacker's Nash equilibrium strategies in some special cases; for example, similar to our case when $V = 0$. Second, it may help the development of efficient methods for computing the Nash equilibrium strategies. Finally, even if the problem exhibits nonlinear convex cost functions, such functions can be approximated well by convex piecewise linear functions.

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Appendix A: A minmax proposition for zero-sum bilinear games

The following records a classic characterization of Nash equilibria for two-person, zero-sum games with bilinear payoffs and with action sets that are polytopes (see Charnes 1953 and Wolfe 1956). It is included for the sake of completeness.

Proposition A *Suppose $X \subseteq R^n$ and $Y \subseteq R^n$ are polytopes and $Q \in R^{m\times n}$. Consider the game where X is the set of options to player* I, *Y is the set of options to player* II, *and upon selection of $x \in X$ and $y \in Y$ the payoff of player* II *to player* I *is $y^\top Qx$. Then*

(a) *There exists a Nash equilibrium and* $\max_{x\in X}\ \min_{y\in Y}\ y^\top Qx = \min_{y\in Y}\ \max_{x\in X} y^\top Qx$.
(b) (x^*, y^*) *is a Nash equilibrium if and only if*

$$x^* \in \operatorname*{argmax}_{x\in X}\Big[\min_{y\in Y}\ y^\top Qx\Big] \tag{20}$$

and

$$y^* \in \operatorname*{argmin}_{y\in Y}\Big[\max_{x\in X}\ y^\top Qx\Big]. \tag{21}$$

(c) *Suppose $X = \{x \in \mathbb{R}^n \mid Ax \le a\}$ and $Y = \{y \in \mathbb{R}^m \mid By \ge b\}$, where $(A, a) \in \mathbb{R}^{p\times n} \times \mathbb{R}^p$ and $(C, b) \in \mathbb{R}^{q\times m} \times \mathbb{R}^q$ (with p and q as positive integers). Then*:
   (i) *x^* satisfies (20) if and only if for some $\lambda^* \in \mathbb{R}^q$, (x^*, λ^*) solves the LP*

$$\begin{aligned} &\max b^\top\lambda\\ &\text{s.t.}\ \ C^\top\lambda - Qx = 0\\ &\qquad Ax \le a\\ &\qquad \lambda \ge 0. \end{aligned} \tag{22}$$

   (ii) *y^* satisfies (21) if and only if for some $\mu^* \in \mathbb{R}^p$, (y^*, μ^*) solves the LP*

$$\begin{aligned} &\min \mu^\top a\\ &\text{s.t.}\ \mu^\top A - y^\top Q = 0\\ &\qquad By \ge b\\ &\qquad \mu \ge 0. \end{aligned} \tag{23}$$

   (iii) *The LP's in (22) and (23) are duals of each other and their common optimal objective value equals* $\max_{x\in X}\ \min_{y\in Y}\ y^\top Qx = \min_{y\in Y}\ \max_{x\in X} y^\top Qx$.

Proof Consider the representation of X and Y given in (c). As X and Y are compact, continuity arguments show that the maxima and minima in (a) are well-defined (and there is no need to use sup's and inf's). Further, standard LP duality shows that for each $x \in \mathbb{R}^n$,

$$\min\big\{y^\top Qx \mid By \ge b\big\} = \max\left\{b^\top\lambda \;\middle|\; \begin{array}{l} C^\top\lambda = Qx\\ \lambda \ge 0 \end{array}\right\}$$

and for each $y \in \mathbb{R}^m$,

$$\max\{y^\top Qx \mid Ax \le a\} = \min\left\{\mu^\top a \;\middle|\; \begin{array}{l}\mu^\top A = y^\top Q\\ \mu \ge 0\end{array}\right\},$$

proving (c). It further follows that the maxmin and minmax of (a) equal the optimal objective values of the LP's in (22) and (23), respectively. As the latter are dual LP's with finite optimal objective function, (a) follows. Finally, part (b) now follows from standard arguments. □

Remark When X and Y are unbounded polyhedra, the maxmin and minmax in part (a) of Proposition A are supinf and infsup, respectively. Still, if one of these expressions is finite, then the sup's and inf's can be replaced by max's and min's, respectively, and the conclusions and proof of Proposition A hold. The next example has unbounded X and Y for which Proposition A does not apply.

Example 3 Let

$$X = \left\{\begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R}\right\}, \qquad Y = \left\{\begin{pmatrix} 1 \\ y \end{pmatrix} : y \in \mathbb{R}\right\} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As $\binom{1}{y}^\top Q\binom{x}{1} = x - y$, we have that the supinf equals $-\infty$ while the infsup equals $+\infty$.

Appendix B: Proof of Proposition 1

Proof (a): Consider $x \in \mathcal{X}(C)$. If $\sum_{j\in M} a_{ij}x_{ij} \le b_i$ for each $i \in N$, then $\hat{\theta}(\pi) \le \sum_{i\in N} \pi_i(b_i - \sum_{j\in M} a_{ij}x_{ij})_+ = \sum_{i\in N} \pi_i(b_i - \sum_{j\in M} a_{ij}x_{ij})$; taking a minimum over x satisfying (7b)–(7d), yields $\hat{\theta}(\pi) \le \theta^*(\pi)$. Next assume that $\hat{x}$ is optimal for (6). Let x' coincide with $\hat{x}$ except that for each i with $b_i - \sum_{j\in M} a_{ij}\hat{x}_{ij} < 0$, the $\hat{x}_{ij}$'s are reduced so that $b_i - \sum_{j\in M} a_{ij}x'_{ij} = 0$. It then follows that $x' \in \mathcal{X}(C)$ and $0 \le b_i - \sum_{j\in M} a_{ij}x'_{ij} = (b_i - \sum_{j\in M} a_{ij}\hat{x}_{ij})_+$ for each $i \in N$. In particular, x' is feasible for (7a)–(7d) and $\hat{\theta}(\pi) = \sum_{i\in N} \pi_i(b_i - \sum_{j\in M} a_{ij}\hat{x}_{ij})_+ = \sum_{i\in N} \pi_i(b_i - \sum_{j\in M} a_{ij}x'_{ij}) \ge \theta^*(\pi)$.

(b): If x is feasible for (7a)–(7d), then it is (trivially) feasible for (6) and (by feasibility for (7c))

$$\sum_{i\in N} \pi_i\left(b_i - \sum_{j\in M} a_{ij}x_{ij}\right)_+ = \sum_{i\in N} \pi_i\left(b_i - \sum_{j\in M} a_{ij}x_{ij}\right).$$

As $\hat{\theta}(\pi) = \theta^*(\pi)$, x is optimal for (6) if and only if it is optimal for (7a)–(7d). □

Appendix C: Proof of Proposition 2

Proof Write LP (7a)–(7d) with equality constraints replacing inequalities by adding non-negative slack variable z_j, for $j \in M$ and s_i, for $i \in N$, to the corresponding m constraints of (7b) and the constraints of (7c), respectively. The coefficient matrix has full rank, and standard results from LP assure that this LP has a basic optimal solution $(x', s, z) \in \mathbb{R}^{n\times m} \times \mathbb{R}^n \times \mathbb{R}^m$ with at most $m + n$ variables that are strictly positive; in particular, x' is optimal for (7a)–(7d). For each $i \in L(x') \cup \nu(x')$ we have $s_i > 0$. As $n = |U(x')| + |L(x')| + |\nu(x')|$, it follows that

$$\left|\left\{(i,j)\in N\times M \mid x'_{ij}>0\right\}\right| = \sum_{i\in N}\left|\left\{j\in M \mid x'_{ij}>0\right\}\right|$$
$$\le m+n-\left|L\left(x'\right)\cup\nu\left(x'\right)\right| \le m+\left|U\left(x'\right)\right|.$$

Further, as $|\{j\in M \mid x'_{ij}>0\}|\ge 1$ for each $i\in U(x')\cup\nu(x')$, it follows that

$$\left|\nu\left(x'\right)\right|+\left|U\left(x'\right)\right| \le \sum_{i\in\nu(x')\cup U(x')}\left|\left\{j\in M \mid x'_{ij}>0\right\}\right|$$
$$=\left|\left\{(i,j)\in N\times M \mid x'_{ij}>0\right\}\right| \le m+\left|U\left(x'\right)\right|.$$

So, $\nu(x')\le m$. □

Appendix D: Proof of Proposition 4

Proof For $x\in\mathcal{X}$ let $N_-(x)\equiv\{i\in N \mid b_i-\sum_{j\in M}a_{ij}x_{ij}<0\}$, and $N_\oplus\equiv N\setminus N_-$.

(a) and (b): Assume that (x^*,w^*) is a Nash equilibrium of the linearized game, i.e.,

$$\max_{w\in\mathcal{W}}\hat{u}_W\left(x^*,w\right)=\hat{u}_W\left(x^*,w^*\right)=\min_{x\in\mathcal{X}}\hat{u}_W\left(x,w^*\right). \tag{24}$$

It follows from the left-hand side of (24) and the explicit expression for $\hat{u}_W(\cdot,\cdot)$ in (8) that $w_i^*=0$ for each $i\in N_-(x^*)$; consequently, $\hat{u}_W(x^*,w^*)=u_W(x^*,w^*)$. As $u_W(x,w)\ge\hat{u}_W(x,w)$ for each $(x,w)\in\mathcal{X}\times\mathcal{W}$, we conclude that for each $x\in\mathcal{X}$

$$u_W\left(x^*,w^*\right)=\hat{u}_W\left(x^*,w^*\right)\le\hat{u}_W\left(x,w^*\right)\le u_W\left(x,w^*\right). \tag{25}$$

Consider any $w\in\mathcal{W}$. Define $\hat{w}=\hat{w}(x^*)\in\mathbb{R}^n$ by

$$\hat{w}_i\left(x^*\right)=\begin{cases} w_i & \text{if } i\in N_\oplus(x^*),\\ 0 & \text{if } i\in N_-(x^*).\end{cases}$$

As $u_W(x^*,w)=\sum_{i\in N}w_i(b_i-\sum_{j\in M}a_{ij}x^*_{ij})_+=\sum_{i\in N}\hat{w}_i(b_i-\sum_{j\in M}a_{ij}x^*_{ij})=\hat{u}_W(x^*,\hat{w})$, the first equality of (24) and $u_W(x^*,w^*)=\hat{u}_W(x^*,w^*)$ imply that

$$u_W\left(x^*,w^*\right)=\hat{u}_W\left(x^*,w^*\right)\ge\hat{u}_W\left(x^*,\hat{w}\right)=u_W\left(x^*,w\right). \tag{26}$$

By (25)–(26), (x^*,w^*) is a Nash equilibrium of the original game, and $V=u_W(x^*,w^*)=\hat{u}_W(x^*,w^*)=\hat{V}$. Of course, $V\ge 0$ as $u_W\ge 0$.

(c): Assume that $V>0$. In view of (a), it suffices to show that a Nash equilibrium (x^*,w^*) of the original game is a Nash equilibrium of the linearized game. As $\sum_{i\in N}w_i^*(b_i-\sum_{j\in M}a_{ij}x^*_{ij})=u_W(x^*,w^*)=\max_{w\in\mathcal{W}}u_W(x^*,w)=V>0$,

$$\emptyset\ne\left\{i\in N: w_i^*>0\right\}\subseteq\operatorname*{argmax}_{i\in N}\left\{b_i-\sum_{j\in M}a_{ij}x^*_{ij}\right\}$$
$$=\left\{i\in N: b_i-\sum_{j\in M}a_{ij}x^*_{ij}=V\right\}; \tag{27}$$

and therefore $\hat{u}_w(x^*,w^*)=\sum_{i\in N}w_i^*V$. Consider $w\in\mathcal{W}$. It follows from $u_W(x^*,w^*)=\sum_{i\in N}w_i^*(V)_+=\sum_{i\in N}w_i^*V=\hat{u}_W(x^*,w^*)$, (4), and $u_W(x^*,w)\ge\hat{u}_W(x^*,w)$ that

$$\hat{u}_W\left(x^*,w^*\right)=u_W\left(x^*,w^*\right)\ge u_W\left(x^*,w\right)\ge\hat{u}_W\left(x^*,w\right). \tag{28}$$

To complete the proof we show that $\hat{u}_W(x^*,w^*)\le\hat{u}_W(x,w^*)$ for each $x\in\mathcal{X}$. To do this, we argue that if $x^*_{ij}>0$ and $w_i^*>0$, then $w_i^*a_{ij}\ge w_s^*a_{sj}$ for each $s\in N$. This inequality is trivial if either $w_s^*=0$ or $a_{sj}=0$. Alternatively, if $w_s^*>0$, $a_{sj}>0$ and $w_i^* a_{ij} < w_s^* a_{sj}$, then (27) and $V>0$ assure that $b_s - \max_{j\in M} a_{ij}x_{ij}^* = V > 0$, and a shift of a small amount of resource j from x_{ij}^* to x_{sj}^* will result in an allocation x' with $s \in N_{\oplus}(x')$ and $u_W(x', w^*) < u_W(x^*, w^*)$, contradicting the first equality in (4). So, for each j, $\{i \in N : w_i^* > 0 \text{ and } x_{ij}^* > 0\} \subseteq \operatorname{argmax}_{i\in N}\{w_i^* a_{ij}\}$; hence, a standard result about the knapsack problem (Dantzig 1957) assures that $x^* \in \operatorname{argmax}_{x\in\mathcal{X}}[\sum_{i\in N}(w_i^* a_{ij})x_{ij}]$. Consequently,

$$x^* \in \operatorname*{argmax}_{x\in\mathcal{X}}\left\{\sum_{j\in M}\sum_{i\in N}(w_i^* a_{ij})x_{ij}\right\}$$
$$= \operatorname*{argmin}_{x\in\mathcal{X}}\left\{\sum_{i\in N} w_i^*\left(b_i - \sum_{j\in M} a_{ij}x_{ij}\right)\right\} = \operatorname*{argmin}_{x\in\mathcal{X}} \hat{u}_W\left(x, w^*\right).$$

□

Appendix E: Proof of Proposition 5

Proof (a) and (b): Except for the equality in (13), (a) follows from Proposition 3(b′), after substituting the explicit expression of $\hat{u}$. Next, (b) follows from the application of Proposition A of Appendix A with

$$Q = \begin{pmatrix} -a_{11} & & & & & & & b_1 \\ & \ddots & & & & & & 0 \\ & & -a_{1m} & & & & & 0 \\ & & & \ddots & & & & 0 \\ & & & & -a_{n1} & & & b_n \\ & & & & & \ddots & & 0 \\ & & & & & & -a_{nm} & 0 \end{pmatrix} \tag{29}$$

(the 0's in the last column of (29) are, respectively, in $\mathbb{R}^{m-2}$, $\mathbb{R}$, $\mathbb{R}^{m\times(n-2)}$, $\mathbb{R}^{m-2}$ and $\mathbb{R}$),

$$X = \left\{ z \in \mathbb{R}^{nm+1} \;\middle|\; \begin{matrix} \begin{pmatrix} z_1 & \cdots & z_n \\ \vdots & \ddots & \vdots \\ z_{(mn-n+1)} & \cdots & z_{mn} \end{pmatrix} \in \mathcal{X} \\ \text{and } z_{nm+1} = 1 \end{matrix} \right\} \quad \text{and}$$

$$Y = \left\{ y \in [0,1]^{mn} \;\middle|\; \begin{matrix} (y_1, y_{m+1}, \ldots, y_{mn-m+1}) \in \mathcal{W} \\ y_i = y_{i-1}, \ \text{for } i \in [nm], \\ \text{with } i (\operatorname{mod} m) \neq 1 \end{matrix} \right\}.$$

Next, the equality in (13) follows from the fact that for each $v \in \mathbb{R}^n$, $\max_{w\in\mathcal{W}} \sum_{i\in N} v_i w_i = \max_{i\in N}(v_i)_+$.

(c): By (a)–(b), there exists $\hat{x} \in \mathcal{X}$ satisfying (13). Let x^* coincide with $\hat{x}$ except that for each i with $b_i - \sum_{j\in M} a_{ij}\hat{x}_{ij} < 0$, the $\hat{x}_{ij}$'s are reduced so that $b_i - \sum_{j\in M} a_{ij}x_{ij}^* = 0$. Then $\max_{i\in N}(b_i - \sum_{j\in M} a_{ij}\hat{x}_{ij})_+ = \max_{i\in N}(b_i - \sum_{j\in M} a_{ij}x_{ij}^*)_+$, implying that x^* satisfies (13) and $b_i - \sum_{j\in M} a_{ij}x_{ij}^* \geq 0$ for each $i \in N$. □

Appendix F: Proof of Lemma 1

Proof (a): Assume that (x', θ^*) is optimal for (11a)–(11d) and $u \equiv \max\{i \in N \mid b_i \geq \theta^*\}$ (as $(0, \theta = b_1)$ is feasible for (11a)–(11d), $b_1 \geq \theta^*$ and therefore u is well-defined). By the feasibility of x' for (11a)–(11d), $b_i - \sum_{j \in M} a_{ij} x'_{ij} \leq \theta^*$ for all $i \in N$. Let x^* be defined in the following way: (i) for $i \leq u$ reduce the x'_{ij}'s so that $\theta^* = b_i - \sum_{j \in M} a_{ij} x^*_{ij}$ while maintaining the nonnegativity (this is possible as $b_i \geq \theta^*$ for $i = 1, \dots, u$), and (ii) or $i > u$, set $x^*_{ij} = 0$ for all $j \in M$. For each $j \in M$ $\sum_{i \in N} x^*_{ij} \leq \sum_{i \in N} x'_{ij} \leq C_j$, so, $x^* \in \mathcal{X}$. Further, since $b_i - \sum_{j \in M} a_{ij} x^*_{ij} = b_i < \theta^*$ for all $i > u$ (following from the definition of u and (5)), (x^*, θ^*) is feasible for (11a)–(11d) and $I(x^*) = [u]$. Also, since (x^*, θ^*) has objective value that equals the optimal one, (x^*, θ^*) is optimal for (11a)–(11d). Following from (the strict version of) (5) we have that $b_n < \cdots < b_{u+1} < \theta^* \leq b_u < \cdots < b_1$ and therefore $\{i \in N \mid b_i = \theta^*\} \in \{\{u\}, \emptyset\}$. For $i = 1, \dots, u-1$, $b_i > \theta^*$ and $\theta^* = b_i - \sum_{j \in M} a_{ij} x^*_{ij}$ imply that $x^*_{ij} > 0$ for some j, and therefore $i \in P(x^*)$. Similarly, if $\theta^* < b_u$, then $u \in P(x^*)$ and $\{i \in N \mid b_i = \theta^*\} = \emptyset$. In the remaining case, $\{u\} = \{i \in N \mid b_i = \theta^*\}$. Thus, as $x^*_{ij} = 0$ for all $i > u$, it follows that $I(x^*) = [u] \subseteq P(x^*) \cup \{i \in N \mid b_i = \theta^*\} \subseteq [u]$. Hence, $P(x^*) \cup \{i \in N \mid b_i = \theta^*\} = [u] = I(x^*)$, in particular, $I(x^*)$ is consecutive. We also conclude that if $\{i \in N : b_i = \theta^*\} = \emptyset$, then $P(x^*) = I(x^*) = [u]$, and alternatively, if $\{i \in N : b_i = \theta^*\} = \{u\}$, then $P(x^*) \in \{[u], [u-1]\}$. So, in either case $P(x^*)$ is consecutive.

(b): Feasibility of (x^*, θ^*) for (11a)–(11d) assures that $\theta^* \geq \hat{\theta} \equiv \max_{i \in N}\{b_i - \sum_{j \in M} a_{ij} x^*_{ij}\}$. Further, the inequality must hold as equality—otherwise, as $\theta^* > 0$, $\hat{\theta}_+ = \max\{\hat{\theta}, 0\} < \theta^*$ and $(x^*, \hat{\theta}_+)$ would be feasible for (11a)–(11d) with objective value $\hat{\theta}_+ < \theta^*$, yielding a contradiction to the optimality of (x^*, θ^*). As $\theta^* = \max_{i \in N}\{b_i - \sum_{j \in M} a_{ij} x^*_{ij}\}$, (15) implies that $I(x^*) = \operatorname{argmax}_{i \in N}\{b_i - \sum_{j \in M} a_{ij} x^*_{ij}\} = \{i \in N : b_i - \sum_{j \in M} a_{ij} x^*_{ij} = \theta^*\}$. Finally, to see that $I^* \equiv \{i \in N : b_i - \sum_{j \in M} a_{ij} x^*_{ij} = \theta^*\} \subseteq P(x^*) \cup \{i \in N \mid b_i = \theta^*\}$, observe that if $v \in I^* \setminus P(x^*)$, then $\theta^* = b_v - \sum_{j \in M} a_{ij} x^*_{ij} = b_v$.

(c): Assume that $a > 0$ and (x^*, θ^*) is optimal for (11a)–(11d). We first prove, by contradiction, that $P(x^*) \subseteq I(x^*)$. So, assume that $P(x^*) \setminus I(x^*) \neq \emptyset$ and $k \in P(x^*) \setminus I(x^*)$; in particular, $k \notin I(x^*)$ and part (b) and feasibility for (11b) assure that $h \equiv \theta^* - [b_k - \sum_{j \in M} a_{kj} x^*_{kj}] > 0$. Also, as $k \in P(x^*)$, there must exist some $q \in M$ with $x^*_{kq} > 0$. Let $\hat{x} \in \mathcal{X}$ coincide with x^* except that x^*_{kq} is decreased by $\epsilon \in (0, \frac{h}{a_{kq}})$ and this quantity is equally distributed to x^*_{iq} for $i \in I(x^*)$. It follows from $a > 0$ and part (b) that

$$
\begin{aligned}
& b_i - \sum_{j \in M} a_{ij} \hat{x}_{ij} \\
&\quad = \begin{cases} b_i - \sum_{j \in M} a_{ij} x^*_{ij} - \frac{a_{iq}\epsilon}{|I(x^*)|} = \theta^* - \frac{a_{iq}\epsilon}{|I(x^*)|} < \theta^* & \text{for } i \in I(x^*), \\ b_k - \sum_{j \in M} a_{kj} x^*_{kj} + \epsilon a_{kq} < \theta^* & \text{for } i = k, \\ b_i - \sum_{j \in M} a_{ij} x^*_{ij} < \theta^* & \text{for } i \in N \setminus (I(x^*) \cup \{k\}). \end{cases}
\end{aligned}
$$

So, $b_i - \sum_{j \in M} a_{ij} \hat{x}_{ij} < \theta^*$ for each $i \in N$ and $\hat{\theta} \equiv (\max_{i \in N}\{b_i - \sum_{j \in M} a_{ij} \hat{x}_{ij}\})_+$ satisfies $0 \leq \hat{\theta} < \theta^*$. As $(\hat{x}, \hat{\theta})$ is feasible for (11a)–(11d) with objective value $\hat{\theta} < \theta^*$, we get a contradiction to the optimality of (x^*, θ^*).

By the above paragraph, if $i \notin I(x^*)$, then $i \notin P(x^*)$ and $\theta^* > b_i - \sum_{j \in M} a_{ij} x^*_{ij} = b_i$, assuring that $b_i \neq \theta^*$. Consequently, $\{i \in N : b_i = \theta^*\} \subseteq I(x^*)$. Combining this conclusion with the above paragraph and part (b), implies that $I(x^*) = P(x^*) \cup \{i \in N : b_i \equiv \theta^*\}$.

We next prove, again by contradiction, that $P(x^*)$ is consecutive. Assume that $u \in N \setminus P(x^*)$ while $(u+1) \in P(x^*)$. As we established that $P(x^*) \subseteq I(x^*)$, $(u+1) \in I(x^*)$. By

$(u+1) \in I(x^*)$ combined with part (b), by $(u+1) \in P(x^*)$ combined with $a > 0$, by the strict version of (5), and by $u \notin P(x^*)$

$$\theta^* = b_{u+1} - \sum_{j \in M} a_{u+1,j} x^*_{u+1,j} < b_{u+1} < b_u = b_u - \sum_{j \in M} a_{uj} x^*_{uj},$$

implying that (x^*, θ^*) is infeasible for (11a)–(11d) and thereby establishing a contradiction. To prove $I(x^*)$ is consecutive, assume for the sake of deriving a contradiction that $u \in N \setminus I(x^*)$ and $u+1 \in I(x^*)$. As we established $P(x^*) \subseteq I(x^*)$, necessarily $u \notin P(x^*)$. It then follows that

$$\theta^* > b_u - \sum_{j \in M} a_{uj} x^*_{uj} = b_u > b_{u+1} \geq b_{u+1} - \sum_{j \in M} a_{u+1,j} x^*_{u+1,j} = \theta^*,$$

establishing a contradiction.

We next prove, again by contradiction, that $\sum_{i \in N} x^*_{ij} = C_j$ for each $j \in M$. Assume that $\sum_{i \in N} x^*_{iq} < C_q$ for some $q \in M$. Let $\hat{x} \in \mathcal{X}$ coincide with x^* except that $\epsilon = C_q - \sum_{i \in N} x^*_{iq}$ is equally distributed among all the x^*_{iq}'s. It then follows from $a > 0$ that for each $i \in N$, $b_i - \sum_{j \in M} a_{ij} \hat{x}_{ij} < b_i - \sum_{j \in M} a_{ij} x^*_{ij} \leq \theta^*$ and therefore $\hat{\theta} \equiv [\max_{i \in N}\{b_i - \sum_{j \in M} x^*_{ij}\}]_+$ satisfies $0 \leq \hat{\theta} < \theta^*$. So, $(\hat{x}, \hat{\theta})$ is feasible for (11a)–(11d) with objective value $\hat{\theta} < \theta^*$, yielding a contradiction to the optimality of (x^*, θ^*).

(d): Assume that (x', θ^*) is optimal for (11a)–(11d). Let $D \equiv \{i \in N \mid b_i \leq \theta^*\}$ and $q \equiv |D|$. Evidently, the x'_{ij}'s for $(i, j) \in D \times M$ can be reduced to 0 without affecting feasibility or optimality to (11a)–(11d). Thus, it can be assumed that $x'_{ij} = 0$ for all $(i, j) \in D \times M$. It follows that constraints (11b) for $i \in D$ and variables x_{ij} for $(i, j) \in D \times M$ can be dropped from LP (11a)–(11d) and each optimal solution of the reduced problem corresponds to an optimal solution of (11a)–(11d) itself (by appropriately adding zero variables). The reduced LP has $n+m-q$ constraints, and $m(n-q)+1$ variables. Adding slack and surplus variables results in a standard form LP with $m(n-q)+1+n+m$ nonnegative variables $n+m-q$ equality constraints whose constraint matrix has full row-rank. This LP has a basic optimal solution, say $(\hat{x}, \hat{\theta} = \theta^*)$, with at most $m+n-q$ nonzero variables, one of which is $\hat{\theta}$. As $b_i > \theta^* = \hat{\theta}$ for $i \in N \setminus D$, the feasibility for (11b) implies that $|\{j \in M \mid \hat{x}_{ij} > 0\}| \geq 1$ for each $i \in N \setminus D$. As $|N \setminus D| + 1 \geq n - q + 1$ positive variables out of at most $m+n-q$ were accounted for, it follows that

$$\sum_{i \in N \setminus D} \left[\left| \{j \in M \mid \hat{x}_{ij} > 0\} \right| - 1 \right] \leq (m+n-q) - (n-q+1) = m-1.$$

Augmenting $\hat{x}$ with the zero variables corresponding to $(i, j) \in D \times M$ yields an optimal solution (x^*, θ^*) of (11a)–(11d) with the properties asserted in (d). □

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