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On Optimality of a Class of Dynamic Myopic Policies for Continuous-Time Replenishment with Periodic Updates

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Abstract This paper is motivated by inventory problems arising in supply chains characterized by continuous replenishment programs based on information exchanged (reviewed) only intermittently between a manufacturing system (supplier) and a customer (retailer). When the replenishment is once-per-period, rather than at any point of time, a well-known result is the optimality of the so-called myopic base-stock policy. We generalize the notion of the base-stock policy and study the optimality of the corresponding class of dynamic myopic policies. We identify a myopic policy and prove that although the replenishment rule is dynamic, this policy is optimal when the demands are stationary and the number of review periods tends to infinity.

Keywords Inventory management · Dynamic programming · Continuous replenishment

1 Introduction

The incorporation of random demand and yield into manufacturing system models has been of interest as early as 1958 [1]. Since then, many authors have considered such problems in various forms. For example, Yano and Lee [2] provided a comprehensive review of the existing literature. Based on system modeling characteristics, they arranged random yield/demand, lot-sizing problems into categories: discrete-time models, which include single-stage models (both single and multiple period); multiple stages in tandem; assembly systems; and continuous-time models with either constant demand rates or random demand rates.

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Classical, multi-period (discrete-time) inventory models are usually treated using recursive dynamic programming; see [3] for a variety of this type of models. When these models incorporate continuous replenishment and continuous inventory updates, they are commonly treated using a continuous-time dynamic programming approach; for example, see [4–7]. If no updates are available during the planning horizon, while the replenishment is continuous, the maximum principle is used [8–10].

This work considers a continuous stochastic control (inventory replenishment) with periodic updates, thereby dealing with the challenge of integrating the above streams of research. The stochastic control problem of inventory replenishment has arisen due to a relatively new approach in allocating responsibility in the replenishment process: vendor-managed inventory (VMI). As opposed to traditional orders, where the customer decides to replenish or not, the VMI approach implies that the supplier, who decides on the customer's behalf, is responsible for the inventory costs incurred by the customer. The decision on how many products to replenish and when is based on the information, which is periodically transferred between the parties. Specifically, the manufacturer handling the retailer's inventories periodically receives updates about the retailer's inventory levels and replenishes them at any point of time [11, 12]. It has been shown that vendor-managed systems lead to better information sharing between the retailer and the manufacturer [13, 14], as well as to increased flexibility in the manufacturer's production operations [15, 16].

The so-called myopic base-stock (order-up-to) policy has been found to be optimal for the classical, once-per-period replenishment system, with end-of-period inventory-related costs when the number of the review periods tends to infinity (see, for example, [3]). Rao, however, analytically shows in [17] that the use of a replenishment interval order-up-to policy under continuous inventory costs, instead of the optimal policy, increases costs by 41.42% in the worst case. Rudi et al. [18] further elaborate on this issue by considering a simple periodic replenishment problem with zero lead time, zero fixed procurement costs and no discounting over an infinite horizon. They investigate the effect of an end-of-period accounting scheme for inventory-related costs, when costs actually accrue in continuous time. They conclude that end-of-period cost accounting without parameter adjustments is, in general, inappropriate and suggest an adjustment to the demand parameters that serves as an approximation.

In this paper, we derive an asymptotically optimal, dynamic, base-stock policy that accounts for continuous inventory costs and continuous replenishment. Specifically, we formulate the continuous replenishment problem (Sect. 2) and generalize the notion of the base-stock policy (Sect. 3). We study the optimality of such a class of myopic policies (Sect. 4) and identify an optimal myopic policy, when costs accrue in continuous time. We prove that, although the optimal replenishment is dynamic, i.e., it depends on time along each review period, this policy is optimal over all possible policies when the number of the review periods tends to infinity (Sect. 5). The results are summarized in Sect. 6.

2 Statement of the Problem

Consider a supplier who replenishes a retailer's inventories with a single-product type. Since the demand for the product is random, the retailer periodically provides the manufacturer with updated information about his inventory levels. Let n be the index of review periods, $n = 1, \dots, N$, each period of length τ. Then period n is determined by time t such that $(n-1)\tau < t \le n\tau$, for $n = 1, \dots, N$. Let the supplier choose a replenishment rate, $u(t)$, which is bounded, i.e.,

$$0 \le u(t) \le U. \tag{1}$$

We assume that sales of period n are lost once the period has been completed, i.e., backlogs are limited to the same period. Although our main results can straightforwardly be extended to account for unlimited backlogs, this realistic assumption facilitates the presentation and enables us to limit the paper to a reasonable length.

Given fluid material flow over a fixed production horizon, $[0, T]$, the retailer's inventory process $X(t)$ is described by the following dynamics:

$$X(t) = \max\{X^{n-1}, 0\} + \int_{(n-1)\tau}^{t} (u(s) - d_n)\,ds$$
$$\text{for } (n-1)\tau \le t < n\tau, n = 1, 2, \dots, N. \tag{2}$$

In (2), X^n is the inventory level at $t = n\tau$ and D_n is the realization of a random demand rate, d_n, at period n. We assume that the demand is stationary, namely, it has the same distribution over all periods. Denote by $f(D_n)$ and $F(x) = \int_{-\infty}^{x} f(D_n)\,dD_n$ the density and cumulative distribution functions of the demand, respectively. Note that, since no new information will become available during a period, the determination along the period of how much to replenish and when must be made based only on the last inventory update (review), X^{n-1}.

The objective is to determine the replenishment rule $\{u(t)|X^{n-1} : (n-1)\tau < t \le n\tau\}$ for each period $n = 1, \dots, N$ over the entire production horizon T in order to minimize the expected average inventory cost

$$J(u, X^0) = \frac{1}{T} E\left[\int_0^T g(X(t))\,dt\right], \tag{3}$$

where $g(\cdot)$ is a piecewise linear cost function

$$g(X(t)) = c^+ X^+(t) + c^- X^-(t), \tag{4}$$

c^+ and c^- are respectively the non-negative unit inventory surplus and backlog costs, $X^+(t) = \max\{0, X(t)\}$, and $X^-(t) = \max\{0, -X(t)\}$.

We let the replenishment plan during period n be $u_n(\cdot)$, i.e., $u_n(\cdot) = u_n(t)$ for $(n-1)\tau < t \le n\tau, n = 1, 2, \dots, N$, and introduce a new notation, $\mathbf{u}^n = [u_n(\cdot), \dots, u_N(\cdot)]$. We next introduce the function

$$J_n(\mathbf{u}^n, X^{n-1}) = \frac{1}{(N-n+1)\tau} E\left[\sum_{i=n}^{N} \int_{(i-1)\tau}^{i\tau} g(X(t))\,dt\right], \tag{5}$$

which is evidently identical to the objective function (3), when $n = 1$, $N\tau = T$. Then the *Bellman* (cost-to-go) function is

$$B_n^{(N)}\big(X^{n-1}\big) = \min_{\mathbf{u}^n}\big\{J_n\big(\mathbf{u}^n, X^{n-1}\big)\big\}, \quad n = 1, \dots, N. \tag{6}$$

Consequently, introducing for convenience

$$G\big(u_n(\cdot), X^{n-1}\big) = \int_{(n-1)\tau}^{n\tau} g\big(X(t)\big)\,dt, \tag{7}$$

the principle of optimality consists of the following recursive dynamic programming equations:

$$B_n^{(N)}(x) = \min_{u_n} \frac{1}{(N-n+1)\tau}\big\{E_n\big[G(u_n(.), x) + (N-n)\tau B_{n+1}^{(N)}(x)\big]\big\},$$
$$n = 1, \dots, N,\ B_{N+1}(x) = 0. \tag{8}$$

The index in the expectation E_n implies that the expectation is taken at period n. The Bellman function is convex and the optimal policy for such a problem is presented in the following proposition (see Theorem 1 in [19]).

Proposition 2.1 *Assuming that the number of periods is finite*, $b = F^{-1}(\frac{c^-}{c^+ + c^-})$, $b < U$, *and that sales are lost at the end of each period*, *the optimal policy at each period is*

$$u_n(t) = \begin{cases} 0, & (n-1)\tau \le t \le t_1, \\ b, & t_1 < t < t_2, \\ 0, & t_2 \le t \le n\tau, \end{cases}$$

where $(n-1)\tau \le t_1 \le t_2 \le n\tau$. *In addition*, *if* $(n-1)\tau < t_1 < t_2 < n\tau$, *then* $t_1 = (n-1)\tau + \frac{X^{n-1}}{b}$.

The calculation of the switching point t_2 is quite convoluted. In the next section we define a myopic policy, which is easy to calculate and implement. Moreover, this policy can become optimal, as will be formulated below.

3 Optimal Myopic Policy for the First Two Periods

We focus on the policies which satisfy two important properties:

– the policies are myopic (as formulated in Definition 3.1 below);
– the inventory level replenished by the end of each period does not change over the periods.

Denote $\inf(d) = \inf\{D \mid f(D) > 0\}$ and $\sup(d) = \sup\{D \mid f(D) > 0\}$. We require that the density function $f(\cdot)$ is continuous in the interval $[\inf d, \sup d)$, and satisfies

$$f(D_n) > 0 \tag{9}$$

for every $D_n, 0 \le \inf d < D_n < \sup d$ and for every n.

Definition 3.1 A policy which depends only on the current review period is referred to as a *myopic* policy.

We extend the notion of base-stock (order-up-to) policy, which is typically employed in discrete-time replenishment problems. Formally, a base-stock policy (with base-stock $Y(n\tau)$ at the end of period n) implies cumulative replenishment of the initial inventory of a period, n, $X^{n-1} \le Y(n\tau)$, along this period so that

$$X^{n-1} + \int_{(n-1)\tau}^{n\tau} u_n(\tau)\, d\tau = Y(n\tau). \tag{10}$$

Thus, with respect to the properties discussed in this section, the myopic policies which we consider are characterized by a constant base-stock level over the periods, denoted by $Y, Y = Y(n\tau), n = 1, \dots, N$, and the replenishment rate at a period independent of future periods. Moreover, according to (10), $Y \ge 0$, since we consider the case of backlogs limited by the end of period, $X^n \ge 0$ and $Y \le b\tau$ (otherwise, $Y > b\tau$ and the replenishment rate must be greater than b at some interval of time, in contradiction to Proposition 2.1). We start off by considering only the first two review periods, $n = 1, 2$. The main results of this section are presented in Theorems 3.1 and 3.2. Specifically, in Theorem 3.1 we prove that there always exists a base-stock, Y^*, in the open interval $(0, b\tau)$, which minimizes the expected cost for the second period ($n = 2$). The proof is based on the first- and second-order optimality conditions (Lemma 3.1 and Remark 3.1, respectively). Theorem 3.2 handles the first period ($n = 1$) showing that the base-stock which minimizes the expected cost for the first period is $b\tau$.

3.1 The Second Period

We assume first that

$$X^0 = 0. \tag{11}$$

Then with respect to a base-stock policy we have $X^0 + \int_0^\tau u_1(\tau)\, d\tau = Y$, i.e., $X^0 \le Y$ for every non-negative Y. In addition, we have $X^1 = \max\{Y - D_1\tau, 0\} \le Y$.

Consider the policy from Proposition 2.1:

$$u_2^*(t) = \begin{cases} 0, & \tau < t \le t_1, \\ b, & t_1 < t \le t_2, \\ 0, & t_2 < t \le 2\tau. \end{cases} \tag{12}$$

Let $t_1 = \tau + \frac{Y(\tau) - D_1\tau}{b}$ and assume that $Y(2\tau) = Y(\tau) = Y$. The switching point, t_2, satisfies

$$Y - D_1\tau + b\left[t_2 - \left(\tau + \frac{Y - D_1\tau}{b}\right)\right] = Y,$$

that is, $t_2 = \tau + \frac{Y}{b}$. In addition, the switching points must satisfy

$$\tau \le t_1 \le t_2 \le 2\tau. \tag{13}$$

Conditions (13) are satisfied, if $Y \le b\tau$, for every $0 \le D_1 \le \frac{Y}{\tau}$.

If $D_1 > \frac{Y}{\tau}$ (i.e., $X^1 = 0$), then, since $t_2 \ge \tau$ for $Y \ge 0$, the policy (12) becomes

$$u_2^*(t) = \begin{cases} b, & \tau < t \le t_2, \\ 0, & t_2 < t \le 2\tau. \end{cases} \tag{14}$$

Hence, for $0 \le Y \le b\tau$, we obtain

$$X^1 + \int_\tau^t u_2 * (s)\,ds = \begin{cases} Y - D_1\tau, & \text{if } 0 \le D_1 \le \frac{Y}{\tau} \text{ and } \tau \le t \le \tau + \frac{Y - D_1\tau}{b}, \\ b(t-\tau), & \text{if } 0 \le D_1 \le \frac{Y}{\tau} \text{ and } \tau + \frac{Y - D_1\tau}{b} < t \le \tau + \frac{Y}{b}, \\ Y, & \text{if } 0 \le D_1 \le \frac{Y}{\tau} \text{ and } \tau + \frac{Y}{b} < t \le 2\tau, \\ b(t-\tau), & \text{if } D_1 > \frac{Y}{\tau} \text{ and } \tau \le t \le \tau + \frac{Y}{b}, \\ Y, & \text{if } D_1 > \frac{Y}{\tau} \text{ and } \tau + \frac{Y}{b} < t \le 2\tau. \end{cases} \tag{15}$$

Denote $\phi_2 = E_1[E_2[G(u_1^*, X^1(Y))]]$, then ϕ_2 depends only on Y for $0 \le Y \le b\tau$.

To find the base-stock, Y^*, which minimizes the expected cost of the second period, we differentiate ϕ_2 with respect to Y:

$$\begin{aligned} \left.\frac{\partial \phi_2(Y)}{\partial Y}\right|_{Y \le b\tau} = {} & \int_0^{\frac{Y}{\tau}} dD_1 \int_\tau^{\tau + \frac{Y - D_1\tau}{b}} dt \bigg[\int_0^{\frac{Y - D_1\tau}{t-\tau}} c^+ f(D_2)\,dD_2 \\ & - \int_{\frac{Y - D_1\tau}{t-\tau}}^\infty c^- f(D_2)\,dD_2 \bigg] f(D_1) \\ & + \int_{\tau + \frac{Y}{b}}^{2\tau} dt \bigg[\int_0^{\frac{Y}{t-\tau}} c^+ f(D_2)\,dD_2 - \int_{\frac{Y}{t-\tau}}^\infty c^- f(D_2)\,dD_2 \bigg]. \end{aligned}$$

Denote by r the ratio between the backlog and surplus costs, $c^- = rc^+$. Consequently, the last expression can be simplified to

$$\begin{aligned} \left.\frac{\partial \phi_2(Y)}{\partial Y}\right|_{Y \le b\tau} = {} & c^+(1+r) \int_b^\infty \frac{f(D_2)}{D_2}\,dD_2 \cdot \int_0^{\frac{Y}{\tau}} (Y - D_1\tau) f(D_1)\,dD_1 \\ & + c^+(1+r) \int_{\frac{Y}{\tau}}^b \frac{Y - D_2\tau}{D_2} f(D_2)\,dD_2. \end{aligned} \tag{16}$$

Remark 3.1 By differentiating ϕ_2 we find that the second derivative is positive:

$$\left.\frac{\partial^2\phi_2(Y)}{\partial Y^2}\right|_{Y\le b\tau} = c^+(1+r)\left[\int_b^\infty \frac{f(D_2)}{D_2}\,dD_2\cdot\int_0^{\frac{Y}{\tau}} f(D_1)\,dD_1 + \int_{\frac{Y}{\tau}}^b \frac{f(D_2)}{D_2}\,dD_2\right] > 0.$$

Consequently the expected cost function is *strictly convex* in $[0, b\tau]$.

In the following lemma we show that expectation ϕ_2 is minimized at an interior point, in between 0 and $b\tau$.

Lemma 3.1 *There exists a point* $Y^* \in (0, b\tau)$, *such that* $\frac{\partial\phi_2}{\partial Y}(Y^*) = 0$.

Proof By substituting $Y = 0$ into (16) we obtain

$$\frac{\partial\phi_2}{\partial Y}(0) = c^+(1+r)\left(-\tau\frac{r}{1+r}\right) = -c^+\tau r < 0. \tag{17}$$

On the other hand, substituting $Y = b\tau$ into (16) we have

$$\frac{\partial\phi_2}{\partial Y}(b\tau) = c^+(1+r)\int_b^\infty \frac{f(D_2)}{D_2}\,dD_2\cdot\int_0^b (b-D_1)\tau f(D_1)\,dD_1 > 0. \tag{18}$$

Thus, according to (17) and (18) (and with respect to the continuity of (16) at $[0, b\tau]$), we find that there always exists $Y^* \in (0, b\tau)$ such that

$$\frac{\partial\phi_2}{\partial Y}(Y^*) = 0. \tag{19}$$

□

From Lemma 3.1 and Remark 3.1 we conclude with the following result.

Theorem 3.1 *Under myopic policies* (12) *and* (14) *with a constant base-stock, there always exists a base-stock level* $0 < Y^* < b\tau$, *which minimizes the expected cost of the second period.*

Corollary 3.1 *The value of the base-stock*, Y^*, *depends on the ratio* r *rather than on the exact values of the unit costs* c^+ *and* c^-.

3.2 The First Period

Given X^0, at the first period, the replenishment policy (Proposition 2.1) is

$$u_1^*(t) = \begin{cases} 0, & 0 \le t \le t_1, \\ b, & t_1 < t \le t_2, \\ 0, & t_2 < t \le \tau, \end{cases} \tag{20}$$

where $t_1 = \frac{X^0}{b}$ and $t_2 = \frac{Y}{b}$ (we must, of course, require that $0 \le X^0 \le Y \le b\tau$). Therefore we have

$$X^0 + \int_0^t u_1 * (s)\, ds = \begin{cases} X^0, & \text{if } 0 \le t \le \frac{X^0}{b}, \\ bt, & \text{if } \frac{X^0}{b} < t \le \frac{Y}{b}, \\ Y, & \text{if } \frac{Y}{b} < t \le \tau \end{cases}$$

and the expected cost is $\phi_1(X^0, Y) := E_1[G(X^0, u_1^*)]$.

In the following theorem we show that the base-stock which minimizes the expected cost of the first period is $b\tau$ (rather than an interior point between 0 and $b\tau$).

Theorem 3.2 *We have*

$$\min_{X^0 \le Y \le b\tau} \phi_1\bigl(X^0, Y\bigr) = \phi_1\bigl(X^0, b\tau\bigr), \tag{21}$$

that is, the base-stock which minimizes the expected cost of the first period is $b\tau$.

Proof Differentiating (21) with respect to Y, we have

$$\frac{\partial \phi_1(X^0, Y)}{\partial Y} = \int_{\frac{Y}{b}}^{\tau} dt \biggl[\int_0^{\frac{Y}{t}} c^+ f(D_1)\, dD_1 - \int_{\frac{Y}{t}}^{\infty} c^- f(D_1)\, dD_1\biggr] \tag{22}$$

(note that this derivative does not depend on X^0).

Substituting $Y = b\tau$, we obtain

$$\left.\frac{\partial \phi_1(X^0, Y)}{\partial Y}\right|_{Y = b\tau} = 0. \tag{23}$$

Differentiating (22) with respect to Y, we also have

$$\frac{\partial^2 \phi_1(X^0, Y)}{\partial Y^2} = \bigl(c^+ + c^-\bigr) \int_{\frac{Y}{b}}^{\tau} \frac{1}{t} f\left(\frac{Y}{t}\right) dt > 0. \tag{24}$$

Thus, $\phi_1(X^0, Y)$ is strictly convex at $[X^0, b\tau]$ and $Y = b\tau$ is a unique optimal base-stock level. □

4 Optimal Myopic Policy for N Periods, $N \ge 2$

In this section we extend our results by proving that the expected cost at period $n, n \ge 3$, is equal to the expected cost at the second period (Lemmas 4.1, 4.2 and Theorem 4.1). Consequently, we show that for any given number of periods, N, there always exists an optimal base-stock, which takes on values in between Y^* and $b\tau$ (Theorem 4.2). Furthermore, the sequence of the optimal base-stocks tends to Y^*, as the number of periods tends to infinity (Theorem 4.3). As a result, the sequence of the corresponding cost-to-go (Bellman) functions converges to a constant limit function (Theorem 4.4) which is proven with the aid of Lemmas 4.3 and 4.4.

Consider N periods (where $N \geq 2$) and let (11) be satisfied. The policy at the first period is again (20), and the expected cost is equal to ϕ_1. Thus, the base-stock which minimizes the expected cost of the first period is equal to $b\tau$. We next consider $2 \leq n \leq N$ and show that the expected cost at period n is equal to that at the second one.

Consider $X^{n-1} = \max\{Y - D_{n-1}\tau, 0\} \leq Y$ for $2 \leq n \leq N$ and $0 \leq Y \leq b\tau$. The policy (12) is then

$$u_n^*(t) = \begin{cases} 0, & (n-1)\tau < t \leq t_1(n), \\ b, & t_1(n) < t \leq t_2(n), \\ 0, & t_2(n) < t \leq n\tau, \end{cases} \tag{25}$$

where $t_1(n) = (n-1)\tau + \frac{Y - D_{n-1}\tau}{b}$ and $t_2(n) = (n-1)\tau + \frac{Y}{b}$. This policy is valid for $0 \leq D_{n-1} \leq \frac{Y}{\tau}$. On the other hand, the policy (14), which corresponds to $D_{n-1} > \frac{Y}{\tau}$, is

$$u_n^*(t) = \begin{cases} b, & (n-1)\tau < t \leq t_2(n), \\ 0, & t_2(n) < t \leq n\tau. \end{cases} \tag{26}$$

Therefore we have

$$\begin{aligned} & X^{n-1} + \int_{(n-1)\tau}^{t} u_n^*(s)\,ds \\ &= \begin{cases} Y - D_{n-1}\tau, & \text{if } 0 \leq D_{n-1} \leq \frac{Y}{\tau} \text{ and} \\ & \quad (n-1)\tau \leq t \leq (n-1)\tau + \frac{Y - D_{n-1}\tau}{b}, \\ b(t-(n-1)\tau), & \text{if } 0 \leq D_{n-1} \leq \frac{Y}{\tau} \text{ and} \\ & \quad (n-1)\tau + \frac{Y - D_{n-1}\tau}{b} < t \leq (n-1)\tau + \frac{Y}{b}, \\ Y, & \text{if } 0 \leq D_{n-1} \leq \frac{Y}{\tau} \text{ and } (n-1)\tau + \frac{Y}{b} < t \leq n\tau, \\ b(t-(n-1)\tau), & \text{if } D_{n-1} > \frac{Y}{\tau} \text{ and } (n-1)\tau \leq t \leq (n-1)\tau + \frac{Y}{b}, \\ Y, & \text{if } D_{n-1} > \frac{Y}{\tau} \text{ and } (n-1)\tau + \frac{Y}{b} < t \leq n\tau \end{cases} \end{aligned}$$

and the expectation (with respect to d_{n-1}) at the current period is

$$\phi_n(Y) := E_{n-1}\big[E_n\big[G\big(u_n^*, X^{n-1}(Y)\big)\big]\big]. \tag{27}$$

Lemma 4.1 *The expectation (27) does not depend on n. Furthermore, for every $n \geq 2$, we have*

$$\phi_n(Y) = \phi_2(Y) \tag{28}$$

for $0 \leq Y \leq b\tau$.

Proof Substituting $t' = t - (n-2)\tau$, $D_n = D_2$ and $D_{n-1} = D_1$ into ϕ_n, we obtain (28). □

For $2 \le n \le N$, we next use two notations:

$$\mathbf{E}_1\big[G\big(u_n^*, X^{n-1}\big)\big] = E_1\Big[E_2\big[\cdots\big[E_{n-1}\big[E_n\big[G\big(u_n^*, X^{n-1}\big)\big]\big]\big]\cdots\big]\Big], \tag{29}$$

$$\varphi_N(Y) = \sum_{n=2}^{N} \mathbf{E}_1\big[G\big(u_n^*, X^{n-1}\big)\big]. \tag{30}$$

Lemma 4.2 *Assume* (11) *holds, then the expectation* (29) *satisfies*

$$\mathbf{E}_1\big[G\big(u_n^*, X^{n-1}\big)\big] = \phi_2(Y) \tag{31}$$

for $2 \le n \le N$.

Proof If $n \ge 3$, then the expectation (29) does not depend on the previous demands $D_1, \dots, D_{n-2}$, and therefore $\mathbf{E}_1[G(u_n^*, X^{n-1})] = E_{n-1}[E_n[G(u_n^*, X^{n-1})]] = \phi_n(Y)$. Thus, with respect to (28), we obtain (31). □

Based on (30) and (31), we conclude with the following theorem.

Theorem 4.1 *For every* $N \ge 2$,

$$\varphi_N(Y) = (N-1) \cdot E_1\big[E_2\big[G\big(u_2^*, X^1\big)\big]\big] = (N-1)\phi_2(Y). \tag{32}$$

Therefore the base-stock which minimizes φ_N is Y^* (note that $\varphi_N(Y)$ does not refer to the first period), and function $\varphi_N(Y)$ is convex at $[0, b\tau]$.

Consequently, we relax assumption (11) as follows:

$$0 \le X^0 \le Y^*, \tag{33}$$

and study the corresponding Bellman (cost-to-go) function.

4.1 The Cost-to-Go Function: Finite Number of Periods

From (6) we obtain that

$$B_1^{(N)}\big(X^0\big) = \min_{\mathbf{u}^1}\big\{J_1\big(\mathbf{u}^1, X^0\big)\big\}. \tag{34}$$

We next extend (30) for $N = 1$ as well, implying that $\varphi_1(Y) = 0$ for $X^0 \le Y \le Y^*$. Consequently, if there is a stationary base-stock, Y, then (34) leads to

$$\begin{aligned} B_1^{(N)}\big(X^0\big) &= \frac{1}{N\tau} \min_{Y \ge X^0}\big\{\phi_1\big(X^0, Y\big) + \varphi_N(Y)\big\} \\ &= \frac{1}{N\tau} \min_{Y \ge X^0}\big\{\phi_1\big(X^0, Y\big) + (N-1)\phi_2(Y)\big\}. \end{aligned} \tag{35}$$

If $N = 1$, then $B_1^{(1)}(X^0) = \frac{1}{\tau}\min_{Y \ge X^0}\{\phi_1(X^0, Y)\} = \frac{1}{\tau}\phi_1(X^0, b\tau)$.

If $N \geq 2$, then, on the one hand, $Y = b\tau$ minimizes $\phi_1(X^0, Y)$, and on the other hand, $Y = Y^*$ minimizes $\varphi_N(Y)$. We now find a *common* base-stock which minimizes the sum $\phi_1(X^0, Y) + \varphi_N(Y)$.

Theorem 4.2 *Let $N \geq 2$. There exists $Y^* < Y_N < b\tau$ such that*

$$B_1^{(N)}(X^0) = \frac{1}{N\tau}\left\{\phi_1\left(X^0, Y_N\right) + \varphi_N(Y_N)\right\}. \tag{36}$$

Proof Denote

$$\tilde{J}^{(N)}\left(X^0, Y\right) = \frac{1}{N\tau}\left[\phi_1(X^0, Y) + \varphi_N(Y)\right] = \frac{1}{N\tau}\phi_1\left(X^0, Y\right) + \frac{N-1}{N\tau}\phi_2(Y).$$

According to (16) and (22), $\tilde{J}^{(N)}$ is continually differentiable with respect to Y at $[Y^*, b\tau]$ (for a given $0 \leq X^0 \leq Y^*$). Consequently,

$$\left.\frac{\partial \tilde{J}^{(N)}(X^0, Y)}{\partial Y}\right|_{Y=Y^*} = \frac{1}{N\tau}\underbrace{\left.\frac{\partial \phi_1(X^0, Y)}{\partial Y}\right|_{Y=Y^*}}_{<0} + \frac{N-1}{N\tau}\underbrace{\left.\frac{\partial \phi_2(Y)}{\partial Y}\right|_{Y=Y^*}}_{=0} < 0$$

and

$$\left.\frac{\partial \tilde{J}^{(N)}(X^0, Y)}{\partial Y}\right|_{Y=b\tau} = \frac{1}{N\tau}\underbrace{\left.\frac{\partial \phi_1(X^0, Y)}{\partial Y}\right|_{Y=b\tau}}_{=0} + \frac{N-1}{N\tau}\underbrace{\left.\frac{\partial \phi_2(Y)}{\partial Y}\right|_{Y=b\tau}}_{>0} > 0.$$

Thus, there exists $Y^* < Y_N < b\tau$, such that $\frac{\partial \tilde{J}^{(N)}(X^0, Y)}{\partial Y}|_{Y=Y_N} = 0$. Since $\tilde{J}^{(N)}$ is convex with respect to Y, then $\tilde{J}^{(N)}(X^0, Y_N) = \min_{Y \geq X^0} \tilde{J}^{(N)}(X^0, Y)$, namely, Y_N satisfies (36). □

4.2 The Cost-to-Go Function: Infinite Number of Periods

Let there be an infinite number of periods. More precisely, we study an optimal solution as the number of periods (N) tends to infinity, and thus consider the *Limit Bellman Function*

$$B_1\left(X^0\right) = \lim_{N \to \infty} B_1^{(N)}\left(X^0\right). \tag{37}$$

We first assume that this limit exists and then show that this assumption holds.

With the stationary base-stock approach, the recursive dynamic programming equation (8) straightforwardly transforms into

$$\begin{aligned} B_1\left(X^0\right) &= \lim_{N \to \infty} \frac{1}{N\tau}\left\{\min_{Y \geq X^0}\left\{\phi_2(Y)\right\} + (N-1)\tau E\left[B_1^{(N-1)}\left(X^0\right)\right]\right\} \\ &= \lim_{N \to \infty} E\left[B_1^{(N-1)}\left(X^0\right)\right], \end{aligned} \tag{38}$$

and by *Lebesgue's dominated convergence theorem*, we obtain

$$B_1(X^0) = E[B_1(X^0)]. \tag{39}$$

When the number of periods increases, the component $\phi_1(X^0, Y)$ in $\tilde{J}^{(N)}(X^0, Y)$ becomes less and less significant. Thus, it is expected that the optimal base-stocks $(Y_N)_{N=1}^{\infty}$ will tend to Y^*. This is formalized as follows.

Theorem 4.3 *If the number of periods tends to infinity, then*

$$\lim_{N\to\infty} Y_N = Y^*. \tag{40}$$

Proof Let $\varepsilon > 0$ and consider the base-stock $Y = Y^* + \varepsilon$. We have $\frac{\partial \phi_2(Y)}{\partial Y}|_{Y=Y^*+\varepsilon} > 0$, and therefore there exists N_0 such that $\frac{\partial \tilde{J}^{(N)}(X^0,Y)}{\partial Y}|_{Y=Y^*+\varepsilon} > 0$ for every $N \geq N_0$ (since $\lim_{N\to\infty} \frac{\partial \tilde{J}^{(N)}(X^0,Y)}{\partial Y} = \frac{1}{\tau}\frac{\partial \phi_2(Y)}{\partial Y}$). Now let $Y > Y^* + \varepsilon$. If $N \geq N_0$ then $\frac{\partial \tilde{J}^{(N)}(X^0,Y)}{\partial Y}|_Y \geq \frac{\partial \tilde{J}^{(N)}(X^0,Y)}{\partial Y}|_{Y^*+\varepsilon} > 0$. However, since $\frac{\partial \tilde{J}^{(N)}(X^0,Y)}{\partial Y}|_{Y=Y_N} = 0$, we proved that $Y_N < Y^* + \varepsilon$. On the other hand, we have $Y_N > Y^*$, and therefore $|Y_N - Y^*| < \varepsilon$ for every $N \geq N_0$. □

Now we can prove that if (33) is satisfied, then the sequence of the Bellman functions converges to a constant limit function, that is, the convergence assumption in (37) always holds.

Lemma 4.3 *Assume that* $0 \leq X^0 \leq Y^*$. *The sequence* $(B_N(X^0))_{N=1}^{\infty}$ *converges at* $[0, Y^*]$. *Furthermore, the limit function is constant and does not depend on the initial condition* X^0.

Proof For every N, the Bellman function satisfies

$$B_1^{(N)}(X^0) = \frac{1}{N\tau}\{\phi_1(X^0, Y_N) + (N-1)\phi_2(Y_N)\}.$$

By the continuity of $\phi_1(X^0, Y_N)$ and $\phi_2(Y_N)$, and with respect to Theorem 4.3, we have

$$\lim_{N\to\infty} \phi_1(X^0, Y_N) = \phi_1(X^0, Y^*) \quad \text{and} \quad \lim_{N\to\infty} \phi_2(Y_N) = \phi_2(Y^*).$$

Therefore we obtain

$$\begin{cases} \lim_{N\to\infty} B_1^{(N)}(X^0) = \lim_{N\to\infty} \frac{1}{N\tau}\{\phi_1(X^0, Y_N) + (N-1)\phi_2(Y_N)\} \\ \qquad = \lim_{N\to\infty}\{\frac{1}{N\tau}\phi_1(X^0, Y_N) + \frac{N-1}{N\tau}\phi_2(Y_N)\} \\ \qquad = \lim_{N\to\infty} \frac{1}{\tau}\phi_2(Y_N) = \frac{1}{\tau}\phi_2(Y^*). \end{cases}$$

Hence, the limit Bellman function does not depend on X^0, and it is determined as

$$B_1 = \frac{1}{\tau}\phi_2(Y^*). \tag{41}$$

□

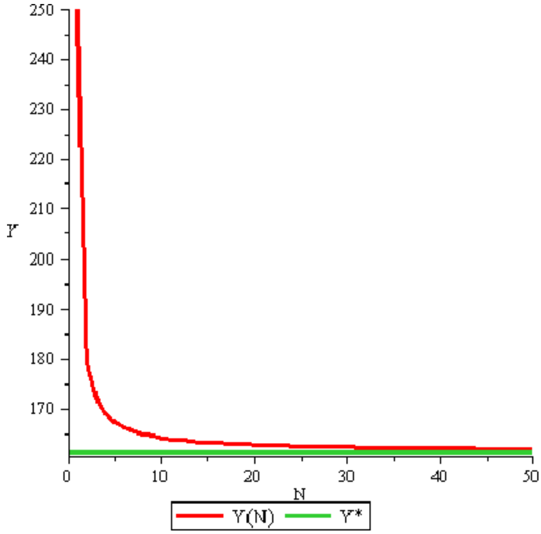

Fig. 1 Optimal base-stock Y_N as N increases for $d_n \sim U[0,50]$, $\tau = 10$, $r = 1$ and $X^0 = 100$

Note that the constant limit Bellman function (41) satisfies the dynamic programming equation (39), that is,

$$B_1 = E[B_1]. \tag{42}$$

Example 4.1 Assume that the distribution of the demand is uniform at $[0,50]$ and that $\tau = 10$, $X^0 = 100$, $r = 1$. Then we have $f(D) = \frac{1}{50}$ and $F(x) = \frac{x}{50}$ for $D, x \in [0,50]$. Therefore $b = 25$ and (16) becomes

$$\left.\frac{\partial \phi_2(Y)}{\partial Y}\right|_{Y \le 250} = 2c^+ \int_{25}^{50} \frac{1}{50 D_2}\, dD_2 \cdot \int_0^{\frac{Y}{10}} \frac{(Y - 10 D_1)}{50}\, dD_1 + 2c^+ \int_{\frac{Y}{10}}^{25} \frac{Y - 10 D_2}{50 D_2}\, dD_2.$$

This expression vanishes when $Y = 161.277$ (that is, $Y^* = 161.277$).

It can be readily seen from Figs. 1 and 2 that the stationary base-stock levels $\{Y_N\}$ become very close to Y^* after only a few periods. The same occurs with the corresponding average costs.

So far we have considered only the cases of $X^0 \le Y^*$. Now we extend Lemma 4.3 for every $X^0 \ge 0$. If $X^0 > Y^*$, then we wait until the inventory is lower than Y^*. Next we replenish one period according to the policy (20) (where X^0 is replaced with the initial inventory of the current period) and all the other periods according to the policies (25) and (26).

Remark 4.1 Assume there exists $n_0 \ge 1$ such that $X^{n_0} \le Y^*$. For every $n \ge n_0$ we have $X^n \le Y^*$, and thus we can use the policies (25) and (26) at those periods. Indeed, if $X^{n_0} \le Y^*$ then $X^{n_0+1} = \max\{Y^* - D_{n_0+1}\tau, 0\} \le Y^*$.

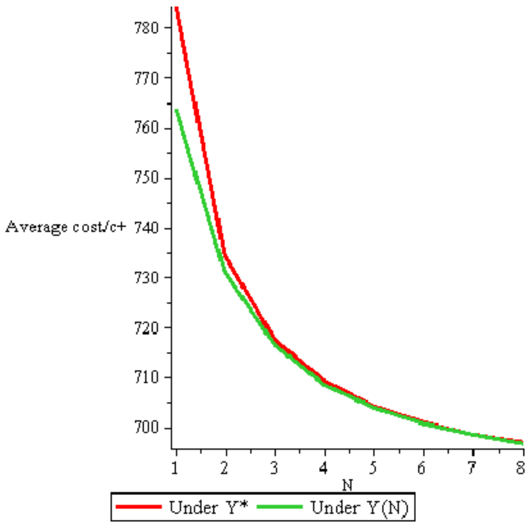

Fig. 2 The expected average costs per period for $d_n \sim U\,[0, 50]$, $\tau = 10$, $r = 1$ and $X^0 = 100$

If the condition in the previous remark is satisfied, then we have the following convergence result.

Lemma 4.4 *If the condition of Remark 4.1 is satisfied, then for any* $X^0 \geq 0$, *the sequence* $(B_N(X^0))_{N=1}^{\infty}$ *converges to the constant limit function* (41).

Proof Let n_0 be the minimal n which satisfies the condition of Remark 4.1 and let $N \geq n_0 + 2$; then

$$\begin{cases} \varphi_N(X^0, Y) = \sum_{n=2}^{N} \mathbf{E}_1\big[G\big(u_n^*, X^{n-1}\big)\big] \\ \qquad\qquad\quad = \sum_{n=2}^{n_0} \mathbf{E}_1\big[G\big(0, X^{n-1}\big)\big] + \mathbf{E}_1\big[G\big(u_{n_0+1}^*, X^{n_0}\big)\big] \\ \qquad\qquad\qquad + \sum_{n=n_0+2}^{N} \mathbf{E}_1\big[G\big(u_n^*, X^{n-1}\big)\big] \end{cases}$$

where $G(0, X^{n-1})$ implies that $u_n(t) = 0$, i.e., nothing is replenished. Here the expectation $\mathbf{E}_1[G(u_{n_0+1}^*, X^{n_0})]$ is equal to $\phi_1(X^{n_0}, Y)$ with respect to (33), and the expectations $\{\mathbf{E}_1[G(u_n^*, X^{n-1})] | n_0 + 2 \leq n \leq N\}$ are equal to $\phi_2(Y)$ (in particular, they do not depend on the previous demands $D_1, \ldots, D_{n_0}$). Hence,

$$\begin{aligned} \varphi_N\big(X^0, Y\big) &= \sum_{n=2}^{N} \mathbf{E}_1\big[G\big(u_n^*, X^{n-1}\big)\big] \\ &= \sum_{n=2}^{n_0} \mathbf{E}_1\big[G\big(0, X^{n-1}\big)\big] + \phi_1\big(X^{n_0}, Y\big) + (N - n_0 - 1)\phi_2(Y) \end{aligned}$$

and the Bellman function (for N periods) is

$$\begin{cases} B_1^{(N)}(X^0) = \frac{1}{N\tau}\min_{Y\ge 0}\{E_1[G(0,X^0)] + \varphi_N(Y)\} \\ \qquad = \frac{1}{N\tau}\min_{Y\ge 0}\{\sum_{n=1}^{n_0}\mathbf{E}_1[G(0,X^{n-1})] + \phi_1(X^{n_0},Y) \\ \qquad\quad +(N-n_0-1)\phi_2(Y)\}. \end{cases}$$

The sum $\sum_{n=1}^{n_0}\mathbf{E}_1[G(0,X^0)]$ does not depend on the base-stock Y and the sum $\phi_1(X^{n_0},Y) + (N-n_0-1)\phi_2(Y)$ is minimized for $Y = Y_{N-n_0}$ (see Theorem 4.2). Furthermore, according to Theorem 4.3, $\lim_{N\to\infty} Y_{N-n_0} = Y^*$. Therefore we obtain

$$B_1^{(N)}(X^0) = \frac{1}{N\tau}\left\{\sum_{n=1}^{n_0}\mathbf{E}_1[G(0,X^{n-1})] + \phi_1(X^{n_0},Y_{N-n_0})\right\} + \frac{N-n_0-1}{N\tau}\phi_2(Y_{N-n_0}),$$

and the limit Bellman function is

$$\lim_{N\to\infty} B_1^{(N)}(X^0) = \lim_{N\to\infty}\frac{1}{\tau}\phi_2(Y_{N-n_0}) = \frac{1}{\tau}\phi_2(Y^*) = B_1.$$

□

It can be shown that if the variance of the demand is finite, then the probability of existence of n_0 which satisfies the condition of Remark 4.1, is equal to 1. Thus we obtain the following theorem.

Theorem 4.4 *Assume that the demand has finite variance. For every initial condition* $X^0 \ge 0$, *the sequence of Bellman functions* $(B_1^{(N)}(X^0))_{N=1}^{\infty}$ *converges to a constant value*, $B_1 = \phi_2(Y^*)$, *with probability one, as the number of periods tends to infinity. Furthermore*, *the optimal base-stock tends to* Y^*, *as the number of periods tends to infinity.*

5 The Optimal Solution for an Infinite Number of Periods

In the previous section we proved that the dynamic myopic policy (20), (25) and (26) with $Y = Y^*$ is optimal for an infinite number of periods when the base-stock Y must be stationary. In this section we prove that the myopic policy is optimal for an infinite number of periods over *all* possible policies (Theorem 5.2). The proof utilizes properties of the discounted cost functions and a relationship between the discounted cost and average cost. Therefore, we next introduce a new objective function, which is the expected *discounted* cost and the corresponding Bellman function:

$$B^{(N,\alpha)}(X^0) = \min_{\mathbf{u}^n}\sum_{n=1}^{N}\mathbf{E}_n[G^{(\alpha)}(u_n,X^{n-1})] = \sum_{n=1}^{N}\mathbf{E}_n[G^{(\alpha)}(u_n^*,X^{n-1})], \qquad (43)$$

where $G^{(\alpha)} = \int_{(n-1)\tau}^{n\tau} e^{-\alpha t} g(X(t))\,dt, \alpha > 0$ and u_n^* is an optimal policy.

Define the limit function

$$B^{(\alpha)}\bigl(X^0\bigr) = \lim_{N\to\infty} B^{(N,\alpha)}\bigl(X^0\bigr). \tag{44}$$

To obtain our main result (Theorem 5.2), we prove that the sequence of the expected average costs $(B_1^{(N)}(X^0))$ converges (Theorem 5.1). Our proof is based on the fact that the limit (44) exists for every $\alpha > 0$ (Lemma 5.1); a function $\eta(\alpha) = \alpha B^{(\alpha)}$ is monotone (Lemma 5.2); and for any given $X^0 \geq 0$, the product $\alpha B^{(\alpha)}(X^0)$ tends to a finite limit as α tends to 0 (Lemma 5.3).

Lemma 5.1 *Function* (47) *is well defined for every* $\alpha > 0$ *and* $X^0 \geq 0$.

Proof Fix $\alpha > 0$ and $X^0 \geq 0$. We have to prove that the sequence $\sum_{n=1}^N (\mathbf{E}_n[G^{(\alpha)}(u_n^*, X^{n-1})])$ converges. Since this sequence is increasing, it is sufficient to show that it is bounded above. Consider the case where the replenishment policy is $u_n(t) = 0$ for every n and every t. There is a probability of one that there exists some n_0 such that $X^{n_0-1} = 0$, and therefore $X^n = 0$ for every $n \geq n_0$. In such a case, we have, for $N \geq n_0$,

$$\sum_{n=n_0}^N \mathbf{E}_n\bigl[G^{(\alpha)}(0,0)\bigr] = \frac{c^-(e^{\alpha\tau} - \alpha\tau - 1)}{\alpha^2} E[d] \sum_{n=n_0}^N e^{-\alpha n\tau}$$
$$< \frac{c^-(e^{\alpha\tau} - \alpha\tau - 1)}{\alpha^2} E[d] \frac{e^{-\alpha n_0\tau}}{1 - e^{-\alpha\tau}} < \infty.$$

Furthermore, for every n we have

$$0 \leq \mathbf{E}_n\bigl[G^{(\alpha)}\bigl(u_n^*, X^{n-1}\bigr)\bigr] \leq \mathbf{E}_n\bigl[G^{(\alpha)}\bigl(0, X^{n-1}\bigr)\bigr].$$

Thus we find that the sequence

$$\sum_{n=1}^N \mathbf{E}_n\bigl[G^{(\alpha)}\bigl(u_n^*, X^{n-1}\bigr)\bigr] = B^{(N,\alpha)}\bigl(X^0\bigr)$$

is bounded above as well , that is, $\lim_{N\to\infty} B^{(N,\alpha)}(X^0)$ exists. □

Note that for every $X^0 \geq 0, \lim_{\alpha\to 0^+} B^{(\alpha)}(X^0) = \infty$. In what follows we prove that the limit

$$\lim_{\alpha\to 0^+} \alpha B^{(\alpha)}\bigl(X^0\bigr) \tag{45}$$

exists. The existence is important since the average costs converge to this limit (divided by τ), as we show below. Specifically, fix X^0 and consider the sequence $\eta_N(\alpha) = \alpha B^{(N,\alpha)}$. According to Lemma 5.1, this sequence converges to a function, $\eta(\alpha) = \alpha B^{(\alpha)}$, for every $\alpha > 0$. In the following lemma we prove that $\eta(\alpha)$ is non-decreasing in (0, 1].

Lemma 5.2 *The limit function $\eta(\alpha)$ is non-decreasing in* $(0, 1]$.

Proof Let $N \geq 1$ and $\alpha \in (0, 1]$. Then for every $1 \leq n \leq N$ the function $\eta_N(\cdot)$ is differentiable in α, and

$$\eta'_N(\alpha) = \sum_{n=1}^{N} \big(1 - \alpha^2\big) \mathbf{E}_n \bigg[\int_{(n-1)\tau}^{n\tau} e^{-\alpha t} g\big(X^*(t)\big)\, dt \bigg] \geq 0. \tag{46}$$

From (46) we conclude that functions $\eta_N(\alpha)$ are non-decreasing in $(0, 1]$, and therefore the limit function $\eta(\alpha)$ is non-decreasing there as well. □

Now we are ready to complete the proof of existence of the limit (45).

Lemma 5.3 *Let $X^0 \geq 0$. The function $\eta(\alpha) = \alpha B^{(\alpha)}(X^0)$ converges to a finite limit, as α tends to zero.*

Proof In accordance with the previous lemma, the function $\eta(\alpha)$ is non-decreasing in $(0, 1]$. In addition, $\eta(\alpha)$ is bounded below by 0. Thus, the (finite) limit (45) exists. □

As mentioned above, the importance of the limit (45) is due to the fact that the expected average costs (multiplied by τ) converge to this limit, as the number of periods tends to infinity. This result is formulated in the following theorem.

Theorem 5.1 *The sequence of the optimal average costs converges. Furthermore, for every $X^0 \geq 0$, we have*

$$\lim_{N \to \infty} B_1^{(N)}\big(X^0\big) = \lim_{\alpha \to 0^+} \frac{1}{\tau} \alpha B^{(\alpha)}\big(X^0\big) \tag{47}$$

(*function $B_1^{(N)}(X^0)$ is defined by* (34)).

Proof Let $X_0 \geq 0$, and consider the superior limit of the sequence of the expected average costs:

$$\begin{cases} \limsup_{N \to \infty} B_1^{(N)}(X^0) = \limsup_{N \to \infty} \frac{1}{N\tau} \sum_{n=1}^{N} \mathbf{E}_n [\int_{(n-1)\tau}^{n\tau} g(X^*(t))\, dt] \\ = \limsup_{N \to \infty} \lim_{\alpha \to 0^+} \dfrac{\sum_{n=1}^{N} \mathbf{E}_n [\int_{(n-1)\tau}^{n\tau} e^{-\alpha t} g(X^*(t))\, dt]}{\tau \underbrace{\int_0^N e^{-\alpha t}\, dt}_{\to N}} \\ = \lim_{\alpha \to 0^+} \limsup_{N \to \infty} \dfrac{[\sum_{n=1}^{N} \mathbf{E}_n \int_{(n-1)\tau}^{n\tau} e^{-\alpha t} g(X(t))\, dt]}{\tau \int_0^N e^{-\alpha t}\, dt} \\ = \lim_{\alpha \to 0^+} \dfrac{\lim_{N \to \infty} \sum_{n=1}^{N} \mathbf{E}_n [\int_{(n-1)\tau}^{n\tau} e^{-\alpha t} g(X(t))\, dt]}{\tau \underbrace{\lim_{N \to \infty} \int_0^N e^{-\alpha t}\, dt}_{= \frac{1}{\alpha}}} = \lim_{\alpha \to 0^+} \frac{1}{\tau} \alpha B^{(\alpha)}(X^0) \end{cases} \tag{48}$$

(see [20] for a formal justification of the interchange in the order of the two limiting operations).

Similarly, we can calculate the inferior limit:

$$\liminf_{N\to\infty} B_1^{(N)}\left(X^0\right) = \lim_{\alpha\to 0^+} \frac{1}{\tau}\alpha B^{(\alpha)}\left(X^0\right). \tag{49}$$

The equality (47) is then immediately obtained from (48) and (49). □

From the convergence of the sequence of the Bellman functions, we readily conclude that the optimal policy for an infinite number of periods is the myopic policy with a stationary base-stock. On the one hand, the limit Bellman function (of the average cost) exists and is well defined. On the other hand, we find that under the myopic policy, the limit Bellman function (41) satisfies the dynamic programming equation (39). Moreover, the Bellman function is convex. Thus, the optimal base-stock (determined by Lemma 3.1) is Y^*. This result is formulated in the following theorem.

Theorem 5.2 *Assuming that the random demands are stationary, their density function is positive in* $(\inf d, \sup d)$ *and their variance is finite. If the number of periods is infinite, then the myopic policy defined by* (20), (25) *and* (26) *with the stationary base-stock* Y^* *is optimal.*

6 Conclusions

In this study we addressed a stochastic, optimal control problem characterized by continuous inventory replenishment, when inventory information is transmitted periodically. The problem, termed vendor-managed inventory, is due to a relatively new approach to the allocation of responsibility in the replenishment process. Assuming that the random demand is stationary, we consider a class of myopic policies with stationary base-stocks at the end of each update period and at most two switching points. The replenishment rate can switch at these points in between zero and a predetermined level, depending on the demand distribution and the ratio between the unit surplus and backlog costs. We prove that if these dynamic policies are applied, there is a unique, optimal base-stock level which depends on the total number of periods. This optimal level converges to a finite limit as the number of periods tends to infinity. We derive a closed-form expression for the limit base-stock level, which depends on the ratio between the inventory backlog and surplus costs.

Based on properties of the discounted cost functions and a study of the relationship between the discounted cost and average cost, we prove that the corresponding cost-to-go functions converge to a constant limit cost-to-go function, which does not depend on the initial condition. As a result, if the variance of the demand is finite, the myopic policy which we determined is the optimal policy over all possible dynamic policies and the optimal base-stock, is the limit base-stock level when the number of periods tends to infinity.

Following [18], we consider a simple setting with zero lead-time and negligible fixed procurement costs. This situation is typical for pharmaceutical supply chains that adopt continuous replenishment programs by allowing deliveries as needed at any point of time. Moreover, the competition between healthcare providers in general and pharmacies specifically has nowadays resulted in increased frequency of deliveries, which may even occur a few times a day. These deliveries are normally carried out with small (low fixed-cost) vehicles. We consider a more general and thereby challenging formulation that would account for fixed costs and lead times as the subject of our future research.

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