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Vertical pricing competition in supply chains: the effect of production experience and coordination

Konstantin Kogan[a] and Charles Tapiero[b]

[a]*Department of Interdisciplinary Studies, Bar-Ilan University, Ramat-Gan, Israel,*
[b]*Department of Finance and Risk Engineering, Polytechnic University of New York, New York, USA*
E-mail: kogank@mail.biu.ac.il [Kogan]

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Abstract

We consider a two-echelon supply chain consisting of a single supplier (producer) and a retailer. The supplier determines the wholesale price with a production cost decreasing with experience. The retailer orders products from the supplier to meet demands. Negative effects of a vertical competition in static supply chain models are typically attributed to a double marginalization. Using an intertemporal supply chain problem, defined by a differential game, we show that in addition to the ‘‘cost’’ of double marginalization, the margin gained from reducing production costs affects the supply chain performance as well. In our analysis, performance is shown to deteriorate even more than the deterioration observed in static problems with no learning (experience). To improve the performance, we provide a time-variant version to the well-known, pure, two-part tariff strategy, which in its dynamic framework may coordinate the supply chain only partially. Efficient coordination in a supply chain is shown to be possible if a mixed two-part tariff strategy is employed, however.

Keywords: supply chains; pricing; production control; differential games

1. Introduction

A retailer’s access to customer information and purchasing behavior as well as its ease in changing prices due to new technologies (including Internet and IT) has motivated extensive research into dynamic pricing in general and continuous-time pricing strategies in particular. Essential attention has been given to (i) dynamic pricing when there are learning effects induced by economies of scale (see, e.g., Kalish, 1983; Jorgenson et al., 1999); (ii) inventory considerations (see the survey by Elmaghraby and Keskinocak, 2003) and (iii) coordinated pricing and production/procurement decisions (see surveys by Chan et al., 2003; Yano and Gilbert, 2002; Cachon, 2003). However,

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despite this broad range of research, relatively few studies are devoted to the continuous-time intra-competition between supply chain parties.

The effect of competition in a static framework on static supply chains and their performance is, however, well studied. Extensive reviews focusing on such competition-related aspects include among others integrated inventory models (Goyal and Gupta, 1989), game theory in supply chains (Cachon and Netessine, 2004), price quantity discounts (Wilcox et al., 1987), and competition/coordination (Leng and Parlar, 2005).

Owing to technical difficulties inherent in continuous-time differential games, the supply chain management literature has been primarily concerned with deterministic models (Cachon and Netessine, 2004; Kogan and Tapiero, 2007). Jorgenson (1986) derives an open-loop Nash equilibrium under static deterministic demand, $d(t) = a(t) - b(t)p(t)$, with price $p(t)$ being a decision variable; demand potential $a(t)$ and customer sensitivity $b(t)$ being constant (and thereby not affecting the supply chain dynamics). Eliashberg and Steinberg (1987) use open-loop Stackelberg solution concepts in a game with a manufacturer and a distributor (both with unlimited capacity) involving quadratic seasonal demand potential $a(t)$ and constant sensitivity $b(t)$. Desai (1992) allows a demand potential to be controlled by an additional decision variable. To address seasonal demands, Desai suggests a numerical analysis for a general case of the open-loop Stackelberg equilibrium for a sine function $a(t)$, constant customer sensitivity $b(t)$ and unlimited manufacturer and retailer capacities (Desai, 1996). To model market interaction with no learning effect, the prices have also been assumed to evolve over time according to some rules (e.g., Sticky Prices) defined thereby as state rather than decision variables. The corresponding games and equilibria are analyzed in Simaan and Takayama (1978), Fershtman and Kamien (1987, 1990), Tsutsui and Mino (1990) and Fibich et al. (2003). For additional applications of differential games in management science and operations research, we refer interested readers to reviews by Feichtinger and Jorgenson (1983), Kogan and Tapiero (2007) and He et al. (2007).

Unlike the studies quoted above which considered the effect of inventories on supply chains, we assume the parties have sufficient capacity and focus on the dynamic effects of production experience on supply chain competition. Further, synchronized firms in supply chains will seek to reduce inventories so that their quantitative importance will be reduced. Thus, our analysis considers a two-echelon supply chain with a supplier—the producer and a retailer who do not cooperate in maximizing their profits but collaborate otherwise in maintaining a synchronized delivery system. The supplier determines the wholesale price while the retailer orders products synchronized with his demand. The demand is assumed to be endogenous in retail price and exogenous in time while the unit production cost decreases as the cumulative production experience increases (learning by doing).

Learning by doing, extensively observed and studied (Thorndike, 1927) has revealed two essential categories (Newell et al., 2001): *power law* and *exponential functions*.

In a power law, learning rates are decreasing and there is no single time scale (Schroeder, 1991). Accordingly, a learning curve is evaluated when cumulative production doubles. This approach has been used extensively to describe the effects of learning by doing as well as the combined effect of experience and other factors (such as technology cost trends). Systematic reviews of this literature can be found in Arrow (1962), Yelle (1979), Dutton and Thomas (1984) and McDonald and Schrattenholzer (2001).

Exponential functions "laws" correspond, however, to a constant learning rate and to a fixed time-scale with learning rate γ, whose inverse defines the intrinsic time-scale of the system. Thus, a plot of the performance logarithm will yield a linear graph with slope γ. Such learning curves were observed in experiments performed by Cohn and Tesauro (1992) with main effects due to learning by doing. These experimental findings then were sustained further by theoretical results of Schuurmans (1997) and Gu and Takahashi (2000). Moreover, it has been shown that under mature technologies (and, thus, low γ) learning curves exhibit a linear behavior in cumulative production (see, e.g., the report by the Australian Business Council for Sustainable Energy, 2003). Such observations thus justify the use of Taylor series expansions in approximating the nonlinear effects of exponential learning by doing curves.

In this paper we also follow the stream of research employing exponential learning and assume in addition mature technologies so that the "exponential learning by doing" can be effectively justified with Taylor series expansions. In the next section (Section 2), we formulate the problem and assume (unlike the references quoted above) that both the demand potential and the customer sensitivity can have a variety of functional forms resulting in a general demand, rather than that specified by a linear price function. We first determine a system-wide optimal solution (Section 3) for a centralized supply chain, which is used as a benchmark and to compare it with a competitive solution. In Section 4 we contrast the static model, which ignores the effects of experience with our inter-temporal, differential game-based approach. We also compare the effects of vertical competition on dynamic pricing and a myopic pricing policy, which ignores the long-run impact of production experience. Finally, facets of supply chain coordination are discussed in Section 5 and insights are provided to mitigate the effects of competition. In particular, we show that in an intertemporal (differential game) framework, a two-part tariff price strategy can be interpreted and implemented in a number of ways. For example, by requiring a fixed payment and setting the wholesale price identical to his current production cost, the supplier may only partially coordinate the supply chain. Greater effect can be achieved if the fixed payment is combined with a mixed strategy, where wholesale prices are chosen randomly around the system-wide optimal production cost. Our results are summarized in Section 6. All proofs are relocated to Appendix A in order to simplify the presentation of our results and conclusions.

2. Problem formulations

Consider a two-echelon supply chain consisting of a single supplier (manufacturer) selling a product type to a single retailer over a period of time, T. The supplier and the retailer have sufficient capacity to deliver in a synchronized manner and process, respectively, the quantity q required at time t. In such an environment, inventory related costs are negligible. We also assume the period during which the parties interact is long enough so that customer's demand (assumed endogenous in the product price), evolve also exogenously over time. Such an assumption justifies the use of Bertrand's model of pricing competition with the quantity sold per time unit, q, depending not only on the product price, p, $\frac{\partial q}{\partial p} < 0$ and $\frac{\partial^2 q}{\partial p^2} \leqslant 0$, but also on the time t elapsed, $q = q(p, t)$ and $\frac{\partial q(p,t)}{\partial t}$ not necessarily equal to zero. An exogenous change in demand is due to the interactions of various factors including seasonal fluctuations, fashion trends, holidays, and customer fatigue. When cumulative sales, $\int_0^t q(p(s), s)\mathrm{d}s$, (i.e., the experience) have little effect on these factors, the dynamic changes can be dealt with in a straightforward manner by setting price adjustments as in traditional and static supply chain models. However, if production (sales) of large quantities result in a learning effect, this will reduce potentially the unit production cost, $c(t)$, then there is a long-term impact of experience that cannot be studied in the framework of static models.

Let the retailer's price per unit be $p(t) = w(t)+m(t)$, where $m(t)$ is the retailer's margin at time t and $w(t)$ is the supplier's wholesale price. Then, if both parties, the supplier and the retailer, do not cooperate to maximize the overall profit of the supply chain over a period T, their decisions, $w(t)$ and $m(t)$, affect each other's revenues at every point of time, resulting in a differential game. In such a game, the supplier chooses a wholesale price, $w(t)$, at each time point t and the retailer selects a margin, $m(t)$ (and thus the quantity $q(p, t)$ ordered at a price $w(t)$ sold to customers at a price $p(t) = w(t)+m(t)$). Consequently, the retailer orders $q(p, t)$ at each time t while the supplier "learns" by its cumulative production, resulting in a reduced unit production cost, $c(t)$. We thus have the following profit maximization problems.

2.1. The supplier's problem

$$\max_{w} J_s(w, m) = \max_{w} \int_0^T (w(t) - c(t))q(w(t) + m(t), t)\mathrm{d}t, \tag{1}$$

s.t.

$$\dot{c}(t) = -\gamma q(w(t) + m(t), t), \ c(0) = C, \tag{2}$$

$$w(t) \geqslant c(t), \tag{3}$$

where γ is the learning rate. Note, that for small γ (of mature technologies assumed in this paper) the general exponential learning curve, $c(t) = c(0)+\mathrm{e}^{-\gamma X(t)}$, where $X(t) = \int_0^t q(p(s), s)\mathrm{d}s$ is the cumulative production, is reduced to $c(t) = c(0)+1-\gamma X(t)$ by a Taylor series expansion. Differentiating this equation we immediately obtain (2). In addition, we assume that the unit production cost cannot become negative for this small γ even under the maximum possible production experience (as is the case in real life), i.e., $C - \gamma \int_0^T q(0, t)\mathrm{d}t \geqslant 0$. Then the sufficient boundary condition is

$$\gamma \leqslant \frac{C}{\int_0^T q(0, t)\mathrm{d}t}.$$

2.2. The retailer's problem

$$\max_{m} J_r(w, m) = \max_{m} \int_0^T m(t)q(w(t) + m(t), t)\mathrm{d}t, \tag{4}$$

s.t.

$$m(t) \geqslant 0, \tag{5}$$

$$q(w(t) + m(t), t) \geqslant 0. \tag{6}$$

Equations (1)–(6) assume a non-cooperative behavior of the supply chain parties which affects the overall supply chain performance. However, when the supply chain is vertically integrated or centralized, so that a single decision-maker is ''in charge'' of the supply chain, the following centralized benchmark problem results.

2.3. The centralized problem

$$\max_{m,w} J(w, m) = \max_{m,w}[J_r(w, m) + J_s(w, m)]$$
$$= \max_{m,w} \int_0^T (w(t) + m(t) - c(t))q(w(t) + m(t), t)\mathrm{d}t, \tag{7}$$

s.t.

(2) – (3) and (5) – (6).

We henceforth omit the independent variable t wherever the dependence on time is obvious.

3. The system-wide optimal solution

To evaluate the best possible performance of the supply chain, we first study the centralized problem by employing the maximum principle. Specifically, the Hamiltonian for the problem (2) – (3), (5) – (6) and (7) is reduced to

$$H(t) = (w(t) + m(t) - c(t))q(w(t) + m(t), t) - \psi(t)\gamma q(w(t) + m(t), t), \tag{8}$$

where the co-state variable $\psi(t)$ is determined by the co-state differential equation

$$\dot{\psi}(t) = -\frac{\partial H(t)}{\partial c(t)} = q(w(t) + m(t), t), \quad \psi(T) = 0. \tag{9}$$

Note that since function (7) is strictly concave, while all constraints are linear, the maximum principle presents not only necessary but also sufficient optimality conditions and the optimal solution which satisfies these conditions is unique.

The Hamiltonian (8) can be interpreted as the instantaneous profit rate, which includes the value $\psi\dot{c}$ of the negative increment in unit production cost created by the economy of scale. The co-state variable ψ is the shadow price, i.e., the net benefit from reducing the unit production cost by one more monetary unit at time t. The differential (9) states that the marginal profit from reducing the production cost at time t is equal to the demand rate at this point. From (9) we have

$$\psi(t) = -\int_t^T q(w(s) + m(s), s)\mathrm{d}s, \tag{10}$$

which implies that $\psi(t) < 0$ for $0 \leqslant t \leqslant T$.

According to the maximum principle, the Hamiltonian is maximized by admissible controls at each point of time. That is, by differentiating (8) with respect to $m(t)$ and $w(t)$ and taking into account that $p(t) = w(t)+m(t)$, we have two identical optimality conditions defined by the following equation

$$q(w(t)+m(t),\, t)+(w(t)+m(t)-c(t)-\psi(t)\gamma)\frac{\partial q(w(t)+m(t),\, t)}{\partial p(t)}=0,$$

where the shadow price (co-state variable) $\psi(t)$ is determined by (10) and the production cost (state variable) $c(t)$ is found from (2)

$$c(t)=C-\gamma\int_0^t q(p,\, s)\mathrm{d}s. \tag{11}$$

Therefore only the optimal price matters in the centralized problem, $p^* \geqslant c$, while the wholesale price, $w \geqslant c$, and the retailer's margin, $m \geqslant 0$, can be chosen arbitrarily so that $p^* = w + m$. This is due to the fact that w and m represent internal transfers of the supply chain. Thus, the proper notation for the centralized payoff function is $J(p)$ rather than $J(m, w)$ and the only optimality condition is

$$q(p^*,\, t)+(p^*-c-\psi\gamma)\frac{\partial q(p^*,\, t)}{\partial p}=0. \tag{12}$$

Explicitly, p^* is the unique optimal price if it satisfies Equation (12) and $p^*(t) \geqslant c(t)$, where c and ψ are determined by (11) and (10), respectively.

Let the maximum price be $P(t)$ at time t, such that $q(P(t)) = 0$. Naturally, assume that $P > c$ and, since, $\psi \leqslant 0$, $P > c + \psi\gamma$. Thus we may verify that if $p - c - \psi\gamma \geqslant 0$, then

$$\frac{\partial^2 H}{\partial p^2}=2\frac{\partial q(p,\, t)}{\partial p}+(p-c-\psi\gamma)\frac{\partial^2 q(p,\, t)}{\partial p^2}<0, \tag{13}$$

and (12) has an interior solution such that $P > p^* \geqslant c + \psi\gamma$. This implies that $p^*(t) > c(t)$ does not necessarily hold at each point of time. At such time points the supply chain is not profitable and the boundary solution $p^*(t) = c(t)$ will be optimal.

Note, that by setting γ at zero we obtain an optimality condition for the corresponding static model which ignores the economy of scale effects:

$$q(p^M,\, t)+(p^M-c)\frac{\partial q(p^M,\, t)}{\partial p}=0. \tag{14}$$

Referring to the static optimal solution p^M at time t as the myopic solution (since it ignores the future learning effect, the long-run effect) and taking into account that $\psi(t) \leqslant 0$ for $0 \leqslant t \leqslant T$, we find that a myopic attitude leads to overpricing.

Proposition 1. *In intertemporal centralized pricing (2) – (3),(5) – (6) and (7), if the supply chain is profitable, i.e., $p > c$, the myopic retail price will be greater than dynamic pricing and the myopic retailers will order less than the system-wide optimal (centralized) price and order quantity respectively for $0 \leqslant t < T$.*

According to Proposition 1, myopic pricing derived from static optimization is not optimal and an intertemporal approach is needed to account properly for the effects of an economy of scale. This does not mean that dynamic optimization necessarily leads to time-dependent prices, however. In what follows we show that if the demand does not explicitly depend on time,

$q(p,t) = q(p)$, the optimal centralized pricing strategy is independent of time. Further, an exogenous increase in demand, for example, results in a price increase, as it is typically the case in monopolistic pricing. This property is stated in the following proposition under the assumption that if $\frac{\partial q(p,t)}{\partial t} < 0$, then $\frac{\partial^2 q(p,t)}{\partial p \partial t} \leqslant 0$ and if $\frac{\partial q(p,t)}{\partial t} > 0$, then $\frac{\partial^2 q(p,t)}{\partial p \partial t} \geqslant 0$.

Proposition 2. *In intertemporal centralized pricing (2) − (3),(5) − (6) and (7), if the supply chain is profitable, i.e., $p > c$, and there is a demand time pattern $q(p,t)$ such that $\frac{\partial q(p,t)}{\partial t}$ exists, then the system-wide optimal price monotonically increases as long as $\frac{\partial q(p,t)}{\partial t} > 0$, and vice versa as long as $\frac{\partial q(p,t)}{\partial t} < 0$. Otherwise, if $\frac{\partial q(p,t)}{\partial t} = 0$ at an interval of time, then the system-wide optimal price and order quantity are constant over the interval.*

4. Intra-competition in supply chain: game analysis

We consider now a decentralized supply chain characterized by non-cooperative firms and assume that both players make their decisions simultaneously. The supplier chooses a wholesale price w and the retailer selects a price, p, or equivalently a margin, m, and hence orders $q(p,t)$ products at each t, $0 \leqslant t \leqslant T$. Since this is a deterministic game, the retailer sells all the products that he has ordered (thus, there are no inventories and orders are synchronized). Using the maximum principle for the retailer's problem, we have

$$H_r(t) = m(t)q(w(t) + m(t),\ t) - \psi_r(t)\gamma q(w(t) + m(t),\ t), \tag{15}$$

where the co-state variable $\psi_r(t)$ is determined by

$$\dot{\psi}_r(t) = -\frac{\partial H_r(t)}{\partial c(t)} = 0,\ \psi_r(T) = 0. \tag{16}$$

Thus, $\psi_r(t) = 0$ for $0 \leqslant t \leqslant T$ and the supplier's production experience does not affect the retailer when parties do not cooperate. That is, myopic pricing is optimal for the non-cooperative retailer. The retailer can in this case simply use the first-order optimality condition to derive a pricing strategy for each time point:

$$\frac{\partial J_r(m,\ w)}{\partial m} = q(w + m,\ t) + m\frac{\partial q(p,\ t)}{\partial p} = 0. \tag{17}$$

It is easy to verify that since the retailer's objective function is strictly concave in m, Equation (17) has a unique solution. Or, by the same token, the retailer's best response function is unique. Comparing (12) and (17), we conclude that if the retailer ignores the long-term dynamic effects of production experience, the supply chain performance deteriorates even more than in the corresponding static case with no learning.

Proposition 3. *In a dynamic vertical competition (or the differential pricing game), myopic pricing is optimal for the retailer. If the retailer and supplier profits at each t, $m > 0$ and $w > c$, the retail price will be greater and the retailer's order will be less than the system-wide optimal (centralized) price* *and order quantity respectively. Moreover, these gaps are even larger than those induced by the corresponding static pricing game.*

Note, that from Proposition 3 and Equation (2) it immediately follows that the system-wide optimal production costs, $c^*(t)$, is lower than the equilibrium production cost for $t > 0$. Moreover, the conclusion that vertical intertemporal pricing competition increases retail prices and decreases order quantities compared to a system-wide optimal solution does not depend on the type of game played. Specifically, it does not depend on whether both players make a simultaneous decision or the supplier first sets the wholesale price and thus plays the role of the Stackelberg leader (see, e.g., Basar and Olsder, 1982). As a result, the overall efficiency of the supply chain deteriorates under intertemporal vertical competition. Furthermore, in addition to the well-known double marginalization effect, we observe the consequences of the learning effect. That is, comparing (12) and (17), we find that deterioration in the supply chain performance is due to the retailer myopically ignoring not only the supplier's margin, $w - c$, from sales at each time point but also the supplier's profit margin from a production cost reduction, $\psi\gamma$. Because of the latter, the deterioration under a competing inter-temporal supply chain is even greater than the one that occurs in the static pricing game, as stated in Proposition 3. Such a difference, however, shrinks over time as the shadow price of cost reduction tends to zero by the end of the product production period T.

To determine the Nash equilibrium which corresponds to the simultaneous moves of the supplier and retailer, we next apply the maximum principle to the supplier's problem. Specifically, we construct the Hamiltonian

$$H_s(t) = (w(t) - c(t))q(w(t) + m(t), t) - \psi_s(t)\gamma q(w(t) + m(t), t),$$

where the co-state variable $\psi_s(t)$ is determined by the co-state differential equation

$$\dot{\psi}_s(t) = q(w(t) + m(t), t), \quad \psi_s(T) = 0. \tag{18}$$

Differentiating the Hamiltonian with respect to wholesale price w we have

$$q(p, t) + (w - c - \psi_s\gamma)\frac{\partial q(p, t)}{\partial p} = 0, \tag{19}$$

which implies that an interior optimal solution determined by (19) is such that $w - c - \psi_s\gamma > 0$. Next, verifying the second derivative of the Hamiltonian, we find that if $w - c - \psi_s\gamma > 0$, then

$$2\frac{\partial q(p, t)}{\partial p} + (w - c - \psi_s\gamma)\frac{\partial^2 q(p, t)}{\partial p^2} < 0.$$

Taking into account that a myopic wholesale price is obtained by setting the learning effect γ at zero, we observe from Equation (19) and the last inequality that (i) the severe problem of double marginalization persists since the supplier ignores the retailer's margin m; (ii) the intertemporal wholesale price is lower than the myopic wholesale price. This implies that the performance of the supply chain further degrades if, in addition to the double marginalization effect, the supplier adopts myopic attitude. Recall that the optimal solution for the retailer is myopic in the intertemporal setting of the decentralized supply chain. It is easy to verify that the supplier's objective function is strictly concave in w and, thus, the supplier's best response (19) is unique as well. Thus, the Nash equilibrium (w^n, m^n) is found by solving simultaneously (19) and (17), which results in

$$w - c - m - \psi_s\gamma = 0 \text{ and } q(c + 2m + \psi_s\gamma,\ t) + m\frac{\partial q(c + 2m + \psi_s\gamma,\ t)}{\partial p} = 0. \tag{20}$$

Note that if the second equation of (20) has a solution in m, then this solution is such that $p = c+2m+\psi_s\gamma > 0$, $w - c - \psi_s\gamma > 0$. Then the second derivative of the second equation of (20) with respect to m yields:

$$3\frac{\partial q(c + 2m + \psi_s\gamma,\ t)}{\partial p} + 2m\frac{\partial^2 q(c + 2m + \psi_s\gamma,\ t),}{\partial p^2} < 0. \tag{21}$$

The latter result does not ensure that $w = c+m+\psi_s\gamma \geqslant c$. We conclude with the following.

Proposition 4. *Let ψ_s be determined by (18), c by (11) and let the dynamic pair (λ, η) be a solution of the system (20) in w and m, respectively. If* $\min\{P - c, \eta\} \geqslant -\psi_s\gamma$, *then the pair $(w^n = \lambda,\ m^n = \eta)$ constitutes a unique open-loop Nash equilibrium of the differential pricing game with $0 \leqslant -\psi_s\gamma < m^n < (P - c - \psi_s\gamma)/2 = P - \lambda$.*

Although, the condition for the Nash equilibrium, $\min\{P - c, \eta\} \geqslant -\psi_s\gamma$, is stated in terms of the co-state variable, a sufficient condition can be obtained by assuming the maximum value for the demand $q(c, t)$, i.e., $\min\{P(t) - c(t), \eta(t)\} \geqslant \gamma \int_t^T q(c, s)\mathrm{d}s$. Note that if c is not replaced with its expression (11), then the solution of Equation (20) at time t becomes a function of the state variable c, and accordingly can be viewed as a closed loop Nash equilibrium. Differentiating both equations of Equation (20), we find that

$$\dot{m}\left[3\frac{\partial q(p,\ t)}{\partial p} + 2m\frac{\partial^2 q(p,\ t)}{\partial p^2}\right] = -\frac{\partial q(p,\ t)}{\partial t} - m\frac{\partial^2 q(p,\ t)}{\partial p \partial t}. \tag{22}$$

Based on Equation (22) we next show that similar to the centralized supply chain, the equilibrium pricing trajectory with respect to the wholesale price and retailer's margin is monotonous under intertemporal competition if the demand time pattern is monotonous. In contrast to the centralized system, where the price p^* barely matters and the only requirement for w and m is $w + m = p^*$, the competition induces a synchronous rate of change of the margins, $\dot{w} = \dot{m}$. This is shown in the following proposition assuming that all conditions of Proposition 4 hold.

Proposition 5. *For the differential pricing game, if the supply chain is profitable, and there is a demand time pattern $q(p,t)$ such that $\frac{\partial q(p,t)}{\partial t}$ exists, then the supplier's wholesale price and the retailer's margin monotonically increase at the same rate as long as $\frac{\partial q(p,t)}{\partial t} > 0$, and they decrease as long as $\frac{\partial q(p,t)}{\partial t} < 0$. If $\frac{\partial q(p,t)}{\partial t} = 0$ at an interval of time, then the Nash equilibrium does not depend on time at the interval.*

To highlight our results, we consider a demand linear in price, $q(p, t) = a(t) - bp$, with $a(t)$ first being an arbitrary function of time. Then we plot the solutions for specific supply chain parameters.

4.1 Example

Let the demand be linear in price with a time-dependent customer demand potential $a(t)$, $q(p,t) = a(t) - bp$, $a > bC$. Because the demand requirements, $\frac{\partial q}{\partial p} = -b < 0$ and $\frac{\partial^2 q}{\partial p^2} = 0$ are met for the selected function, we employ Proposition 4 to solve the system (20), which, for a linear demand, takes the following form:

$$a - b(c + 2m^n + \psi\gamma) - bm^n = 0, \tag{23}$$

$$w^n = c + m^n + \psi\gamma. \tag{24}$$

Using Equation (22) or, equivalently, by differentiating (23) and (24) we have $\dot{w}^n = \dot{m}^n = \frac{\dot{a}}{3b}$ and thus:

$$m^n(t) = m^n(T) - \frac{a(T)}{3b} + \frac{a(t)}{3b}, \quad w^n(t) = w^n(T) - \frac{a(T)}{3b} + \frac{a(t)}{3b}.$$

In addition, we obtain from (23) $a(T) - bc^n(T) - 3bm^n(T) = 0$. Thus, $m^n(T) = \frac{a(T)}{3b} - \frac{c^n(T)}{3}$. However, according to (24), we have $w^n(T) = c^n(T) + m^n(T)$, that is, $w^n(T) = \frac{a(T)}{3b} + \frac{2c^n(T)}{3}$. Substituting m^n and w^n, we find

$$c^n(T) = C - \gamma \int_0^T \left\{ a(t) - b\left[\frac{2a(t)}{3b} + \frac{c^n(T)}{3}\right]\right\} \mathrm{d}t, \tag{25}$$

which results in

$$c^n(T) = \frac{3C - \gamma A(T)}{3 - \gamma bT}, \tag{26}$$

where $A(T) = \int_0^T a(t)\mathrm{d}t$.

We next follow our assumption that the system parameters are such that the terminal production cost, $c^n(T)$, is positive, no matter how experienced the manufacturer becomes, i.e., $\gamma bT < 3$ and $3C > \gamma A(T)$. Consequently, if $\frac{a(t)}{3b} \geqslant \frac{3C - \gamma A(T)}{3(3 - \gamma bT)}$, then the Nash equilibrium of the differential pricing game under linear in price demand is

$$w^n(t) = \frac{a(t)}{3b} + \frac{2(3C - \gamma A(T))}{3(3 - \gamma bT)} \text{ and } m^n(t) = \frac{a(t)}{3b} - \frac{3C - \gamma A(T)}{3(3 - \gamma bT)}, \tag{27}$$

Otherwise, at least one of the parties is not always profitable and the equilibrium involves boundary solutions at some intervals of time. Next, the overall price, $m^n + w^n$, that the retailer charges and the quantity he orders are

$$p^n(t) = \frac{2a(t)}{3b} + \frac{3C - \gamma A(T)}{3(3 - \gamma bT)} \text{ and } q^n(t) = \frac{a(t)}{3} - \frac{3C - \gamma A(T)}{3(3 - \gamma bT)}b, \tag{28}$$

respectively. To find the system-wide optimal solution (12), determined for the linear demand function by the equation

$$a - bp^* - (p^* - c - \psi\gamma)b = 0, \tag{29}$$

we first differentiate it with respect to time to obtain $\dot{p}^* = \frac{\dot{a}}{2b}$. Then from (29) we have the terminal boundary condition $a(T) + bc(T) - 2bp^*(T) = 0$, that is, $\frac{a(T)}{2b} + \frac{c(T)}{2} = p^*(T)$. Thus, $p^* = \frac{a}{2b} + \frac{c(T)}{2}$. Substituting found centralized solution into (2) we have

$$c(T) = C - \gamma \int_0^T a(t) - b\left(\frac{a(t)}{2b} + \frac{c(T)}{2}\right)\mathrm{d}t, \tag{30}$$

which results in

$$c(T) = \frac{2C - \gamma A(T)}{2 - \gamma bT}. \tag{31}$$

Comparing (25) and (3) and taking into account that $a > bC$, we observe that even if the terminal production costs in the right-hand side of these equations are identical $c(T) = c^n(T)$, the Nash cost $c^n(T)$ in the left-hand side of (25) is greater than $c(T)$ for the centralized case (Equation (30)). Consequently, assuming that $\gamma bT < 2$ implies $\gamma b < 1$ and we have, when comparing (26) and (31),

$$\frac{2C - \gamma A(T)}{2 - \gamma bT} < \frac{3C - \gamma A(T)}{3 - \gamma bT}. \tag{32}$$

Then the system-wide optimal price that the retailer charges his customers and the quantity he orders are

$$p^*(t) = \frac{a(t)}{2b} + \frac{2C - \gamma A(T)}{2(2 - \gamma bT)} \text{ and } q^*(t) = \frac{a(t)}{2} - \frac{2C - \gamma A(T)}{2(2 - \gamma bT)}b. \tag{33}$$

Using inequality (32), one can immediately observe that both terms of the price-defining equation (33), $\frac{a(t)}{2b}$ and $\frac{2C - \gamma A(T)}{2(2 - \gamma bT)}$, are smaller than the corresponding terms of the Nash price in Equation (28), as stated in Proposition 3.

4.2. A numerical application

In what follows, we illustrate with Maple the Nash solution (28) for specific parameters of the differential pricing game. Let the demand potential $a(t)$ be exponentially decreasing over time, $a(t) = 10e^{-0.1t}$. Other system parameters are: $b = 0.1$, $C = 11$, $T = 8$, $\gamma = 0.05$. We first define the potential $a(t)$ and its cumulative value $A(T)$ with Maple:

$$a := 10e^{(-0.1t)}$$

$$A := 100. - 100.e^{(-0.1000000000\,T)}.$$

Next we determine the Nash wholesale price w; margin m; price p; system-wide optimal price p^*; quantity q; shadow price (co-state variable) ψ; and production cost (state variable) c:

$$w := \frac{10}{3}\frac{e^{(-0.1\,t)}}{b} + \frac{2\left(3C - \gamma\left(100. - 100.e^{(-0.1000000000\,T)}\right)\right)}{9 - 3\gamma bT}$$

$$m := \frac{10}{3}\frac{e^{(-0.1\,t)}}{b} - \frac{3C - \gamma\left(100. - 100.e^{(-0.1000000000\,T)}\right)}{9 - 3\gamma bT}$$

$$p := \frac{20}{3}\frac{e^{(-0.1\,t)}}{b} + \frac{3C - \gamma\left(100. - 100.e^{(-0.1000000000\,T)}\right)}{9 - 3\gamma bT}$$

$$q := \frac{10}{3}e^{(-0.1\,t)} - \frac{\left(3C - \gamma\left(100. - 100.e^{(-0.1000000000\,T)}\right)\right)}{9 - 3\gamma bT}$$

$$\begin{aligned}\psi :=& 0.3333333333\,\Big(300.e^{(-0.1000000000\,t)} - 57.721256649\,e^{(-0.1000000000\,t)} b\,T + 3.t\,C\,b \\ & - 57.72156649\,b\,t + 57.72156649\,b\,t\,e^{(-0.1000000000\,T)} - 300.e^{(-0.1000000000\,T)} \\ & - 3.C\,b\,T + 57.72156649\,b\,T\Big)/(-3. + 0.5772156649\,b\,T)\end{aligned}$$

$$\begin{aligned}c :=& C - 0.1924052216\,\Big(-300. + 57.72156649\,b\,T + 300.e^{(-0.1000000000\,t)} \\ & - 57.72156649\,e^{(-0.1000000000\,t)}\,b\,T + 3.b\,t\,C - 57.72156649\,b\,t \\ & + 57.72156649\,b\,t\,e^{(-0.1000000000\,T)}\Big)/(-3. + 0.5772156649\,b\,T)\end{aligned}$$

$$p^* := 50.e^{(-0.1\,t)} + 4.336734694 + 1.275510204e^{(-0.8000000000)}.$$

Finally, we substitute the chosen system parameters into the Nash equations and plot the results (see Figs 1–3).

In Fig. 1 we observe that the Nash retail price is higher than the system-wide optimal price and thus the Nash demand (the quantity of products ordered and sold) is lower than the system-wide optimal demand, as shown in Proposition 3.

5. Coordination

As shown previously, the negative effects of an intertemporal vertical competition is due to a double marginalization persistent at each time point as in static models and to a dynamic learning effect in economy of scale. It is thus the learning effect that induces a new margin compared with the corresponding static pricing game. In contrast to margins from sales, this new margin is gained from a production cost reduction. Thus, deterioration takes place if the retailer ignores both the supplier's profit margin, $w - c$, and the supplier's margin from cost reduction, $-\psi_s\gamma$.

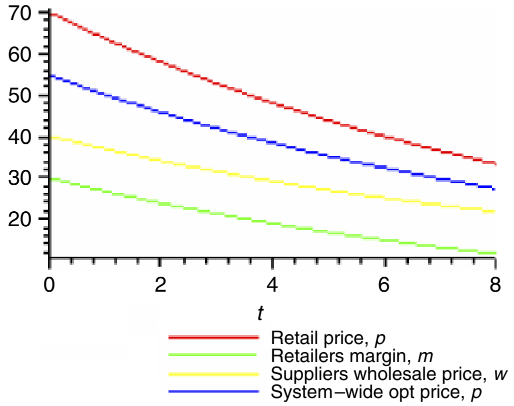

Fig. 1. Nash equilibrium: retail price, retailer's margin and supplier's wholesale price versus system-wide optimal retail price.

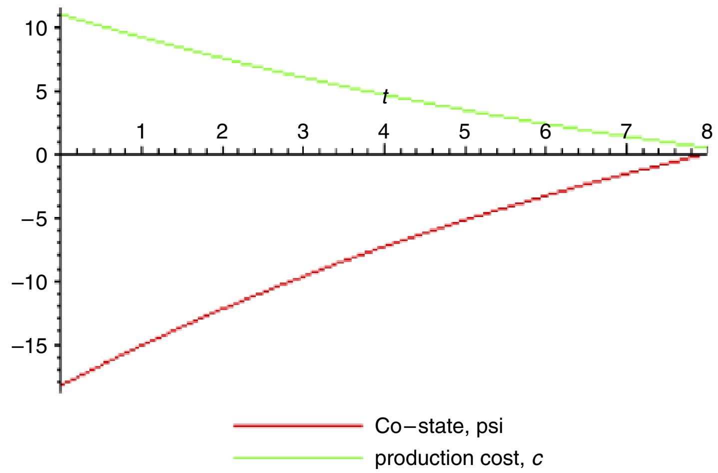

Fig. 2. Nash equilibrium: evolution of the shadow price and production cost.

Specifically, recalling that $p = w+m$, the retailer's best response is

$$q(p,\, t) + m\frac{\partial q(p,\, t)}{\partial p} = 0,$$

which implies that, although the demand depends on two margins, $w+m$, and the supplier has an added margin resulting from cost reduction, the retailer takes into account only his margin m rather than ordering with respect to the centralized approach (12)

$$q(p^*,\, t) + (p^* - c - \psi\gamma)\frac{\partial q(p^*,\, t)}{\partial p} = q(p^*, t) + (w^* + m^* - c - \psi\gamma)\frac{\partial q(p^*,\, t)}{\partial p} = 0$$

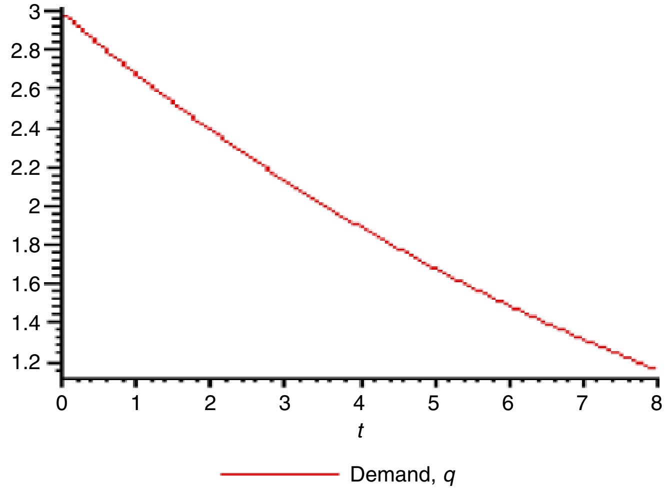

Fig. 3. Nash equilibrium: evolution of the demand.

and thus, adding the supplier's margins $w-c$ and $-\psi_s\gamma$ to m. At the same time, from Equation (19) we observe that the supplier ignores the only margin, m, which the retailer has when setting the wholesale price.

For a producer, an essential question is, of course, how to induce the retailer to order more, while for the retailer the question is how to induce the supplier to reduce the wholesale price. These issues pertain to how we coordinate the supply chain and increase its profits. It turns out in our analysis that the two-part tariff approach widely employed in static supply chain approaches coordinates supply chains functioning in dynamic conditions as well. However, in contrast to static models, the two-part tariff allows for the implementation of different strategies, which do not necessarily result in perfect coordination. That is, an optimal solution under vertical competition may not converge to the system-wide optimal solution. Specifically, if the supplier is the leader, he can set the wholesale price equal to his production cost, but charge the retailer with a fixed (possibly time-dependent) fee. With this dynamic version of the two-part tariff strategy, the supplier induces the retailer to order more products and regulates his share in the total supply chain profit without a special contract.

To show the effect of the dynamic two-part tariff on the supply chain, let the supplier be a leader who first sets the wholesale price $w(t) \equiv c(t)$, then the Hamiltonian of the retailer's problem takes the following form

$$H(t) = m(t)q(c(t)+m(t),\, t) - \psi_r(t)\gamma q(c(t)+m(t),\, t), \tag{34}$$

where the co-state variable $\psi_r(t)$ is determined by

$$\dot{\psi}_r = (\psi_r\gamma - m)\frac{\partial q(c+m,\, t)}{\partial p},\ \psi(T) = 0. \tag{35}$$

Then the margin the retailer sets is found by differentiating the Hamiltonian with respect to m,

$$q(c+m,\,t)+(m-\psi_r\gamma)\frac{\partial q(c+m,\,t)}{\partial p}=0. \tag{36}$$

Comparing (12) and (36) we observe that the retailer orders a system-wide optimal quantity if both $w(t)\equiv c(t)$ and $\psi_{\mathrm{r}}\equiv\psi$ hold (the retailer's shadow price is identical to the system-wide shadow price which, with respect to (35) and (9), cannot hold). In other words, the retailer accounts for a learning effect with a shadow price $\psi_r<0$ instead of $\psi<0$. Accordingly, by setting $w(t)\equiv c(t)$ and charging fixed fees for orders, the supplier eliminates double marginalization and also induces the retailer to partially take into account the supplier's margin from cutting the production cost. However, the optimal retailer's response will never be equal to the system-wide optimal solution. The explanation of the dynamic two-part tariff's partial efficiency is due to the dynamics of the cumulative memory. Repeated setting of the marginal price to $w(t)=c(t)$ during a period of time, transforms the decision or control variable $w(t)$ into a state variable, identical to the state variable $c(t)$ whose dynamic properties are known and can thus be accounted for by the retailer. This is in contrast to memoryless static models, which do not account for previous settings or future effects.

We also note that since $\psi_r(T)=\psi(T)=0$, ψ tends to ψ_r over time. This implies that time has a coordinating effect on the supply chain which becomes perfectly coordinated with the dynamic two-part tariff by the end of the production period. This passive way, however, is not the only way to coordinate and improve the supply chain profit with a two-part tariff. An alternative approach consists in setting the wholesale price equal to the system-wide, time-dependent, production cost, $w(t)\equiv c^*(t)$. This time-variant two-part tariff strategy implies that the wholesale price is only a function of time rather than only of the learning experience. Consequently, $w(t)$ remains a decision variable and the supply chain can be perfectly coordinated. The disadvantage of this two-part tariff price, however, is that since the wholesale price will follow exactly the evolution of the supplier production cost, the retailer may still interpret it as the dynamic two-part tariff. Consequently, the retailer may deviate from the system-wide optimal order quantity at some point in time. To prevent this type of time-inconsistency, the supplier may choose another type of two-part tariff strategy. For example, the supplier, instead of choosing pure strategies with either $w(t)\equiv c(t)$ or $w(t)\equiv c^*(t)$, may employ a mixed two-part tariff strategy. With such a strategy, the wholesale price could be selected randomly at constant levels around the production cost $c^*(t)$ over some fixed intervals of time. In such a case, $w(t)$ is announced as a deterministic function of time, $\hat{w}(t)$, rather than of the learning dynamics or of the optimal production cost and the retailer's optimality condition is reduced to

$$q(\hat{w}+m,\,t)+(m-\psi_r\gamma)\frac{\partial q(\hat{w}+m,\,t)}{\partial p}=0.$$

As long as wholesale prices $\hat{w}(t)$ are not affected by the demand experience, the closer the price $\hat{w}(t)$ to $c^*(t)$, the more coordinated the supply chain will be. Further, the risk of viewing this strategy as a pure dynamic two-part tariff will also be reduced.

6. Conclusions

In this paper we develop a differential game to model continuous-time dynamic interactions between a supplier and a retailer who faces a general, endogenous in product price and exogenous in time demand. The dynamic is due to the exponential learning by doing in mature technologies. With such an approach we address the problem of vertical competition in a supply chain with experience accumulating and impacting production costs over time. We have shown that a myopic attitude based on static models of supply chains leads to overpricing. This observation justified therefore the intertemporal approach used in this paper and needed to account properly for the economy of scale effects. For comparative purposes, we have studied both a centralized (or vertically integrated) and a non-cooperating supply chain. In contrast to the centralized system, where the retail price barely matters, we note in our analysis that competition induces not only higher pricing, but also a synchronized rate of change in the wholesale price and the retailer's margin. Furthermore, if the long-term dynamic effects of production experience are disregarded, the supply chain performance is shown to deteriorate even more than in the corresponding static case with no learning (experience). Such deterioration in supply chain performance is due to the retailer myopically ignoring not only the supplier's margin from sales at each time point but also the supplier's profit margin from production cost reduction.

An essential result outlined in our analysis is that the pure two-part tariff strategy widely employed in static supply chain approaches may only partially coordinate supply chains in a dynamic setting. This effect is due to a cumulative "memory" or time experience imbedded in the continuous time exchange between the retailer and the supplier. Although, time has a coordinating effect on the supply chain as well, it becomes perfectly coordinated with the dynamic two-part tariff only by the end of the production period. A pure time-variant strategy, setting the wholesale price equal to the system-wide optimal production cost may still be interpreted as dynamic and, as a result, become time-inconsistent on the other hand.

To compensate this time inconsistency, our analysis has suggested that the supplier may employ a mixed two-part tariff strategy with the wholesale price selected randomly around the production cost over some fixed intervals of time. In such a case, the wholesale price will be a function of time, rather than learning. Accordingly, the closer the wholesale price to the production cost, the more coordinated the supply chain will be. While the paper has emphasized only the experience gained in production, other sources of system dynamics such as customer and brand loyalty, word of mouth and risk aversion are cases for further study where additional experience effects can be tracked to a supply chain performance. Studying these effects in their time setting on supply chain performance is a challenging task, which presents possible directions for future research.

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APPENDIX

Proof of Proposition 1: Comparing (14) and (12) we observe that

$$q(p^*, t) + (p^* - c - \psi\gamma)\frac{\partial q(p^*, t)}{\partial p} = q(p^M, t) + (p^M - c)\frac{\partial q(p^M, t)}{\partial p} = 0, \tag{A.1}$$

while taking into account that $p > c$, $\psi < 0$ for $0 \leqslant t \leqslant T$, and $\frac{\partial q}{\partial p} < 0$,

$$q(p^M, t) + (p^M - c - \psi\gamma)\frac{\partial q(p^M, t)}{\partial p} < q(p^M, t) + (p^M - c)\frac{\partial q(p^M, t)}{\partial p} = 0. \tag{A.2}$$

Next, by denoting $f(p) = q(p, t) + (p - c - \psi\gamma)\frac{\partial q(p,t)}{\partial p}$, and recalling (13), we conclude that $\frac{\partial f(p)}{\partial p} < 0$.

Thus, from conditions (A.1) and (A.2) we have $f(p^M) < f(p^*)$, which with respect to the last inequality requires that $p^M > p^*$ and, hence, $q(p^M) < q(p^*)$, as stated in Proposition 1. ■

Proof of Proposition 2: Differentiating (12), we have

$$\frac{\partial q(p^*, t)}{\partial t} + \frac{\partial q(p^*, t)}{\partial p}\dot{p}^* + (p^* - c - \psi\gamma)\left[\frac{\partial^2 q(p^*, t)}{\partial p^2}\dot{p}^* + \frac{\partial^2 q(p^*, t)}{\partial p \partial t}\right] + \dot{p}^*\frac{\partial q(p^*, t)}{\partial p}$$

and thus

$$\dot{p}^*\left[2\frac{\partial q(p^*, t)}{\partial p} + (p^* - c - \psi\gamma)\frac{\partial^2 q(p^*, t)}{\partial p^2}\right] = -\frac{\partial q(p^*, t)}{\partial t} - (p^* - c - \psi\gamma)\frac{\partial^2 q(p^*, t)}{\partial p \partial t}.$$

Recalling the assumption and (13) we readily observe that $\dot{p}^* > 0$ if $\frac{\partial q(p^*, t)}{\partial t} > 0$, otherwise, $\dot{p}^* \leqslant 0$. ■

Proof of Proposition 3: The first statement is due to the fact that $\psi_r = 0$. Comparing (12) and (17) and taking into account that $p = w + m$, $m > 0$ and $w > c$, $\psi < 0$ for $0 \leqslant t < T$, and $\frac{\partial q}{\partial p} < 0$, we observe that

$$q(p^*) + (p^* - w)\frac{\partial q(p^*)}{\partial p} > q(p^*) + (p^* - c - \psi\gamma)\frac{\partial q(p^*)}{\partial p} = 0.$$

This inequality holds even under the myopic attitude, $\psi = 0$, which implies that the deterioration in the supply chain performance increases (the third statement of the proposition), when it is affected by the economy of scale. Since the second derivative of both sides of the inequality is negative, the remaining part of the proof is similar to that for Proposition 1. ■

Proof of Proposition 4: To see that a solution of (20) exists and that it is unique, assume $m^n = 0$ at a point t. Then, since $P(t) > c(t) + \psi_s(t)\gamma$ and $q(P) = 0$, $q(c + 2m^n + \psi_s\gamma, t) > 0$, while the second term in the second equation of (20) is zero.

Using notation of $f(m^n)$ for the left-hand side of the second equation of (20), we thus found that $f(m^n) = q(c + m^n + \psi_s\gamma,\ t) + m^n \frac{\partial q(c+m^n+\psi_s\gamma,\ t)}{\partial p} > 0$, when $m^n = 0$. On the other hand, by letting $c + 2m^n + \psi_s\gamma = P$ and accounting for the fact that $q(P, t) = 0$, $m^n = (P - c - \psi_s\gamma)/2 > 0$ and that as a result, the second term of the second equation of (20) is strictly negative, we observe that $f(m^n) < 0$. Consequently, taking into account that $\frac{\partial f(m^n)}{\partial m^n} < 0$, we conclude that the solution of $f(m^n) = 0$ is unique and meets the following condition $0 < m^n < (P - c - \psi_s\gamma)/2$.

Finally, requiring $m^n \geqslant -\psi_s\gamma$ and $(P - c - \psi_s\gamma)/2 > -\psi_s\gamma$, i.e., $min\{P - c, \eta\} \geqslant -\psi_s\gamma$, we readily verify that the first equation of (20), $w = c + m + \psi_s\gamma$, always has a unique feasible solution as well. ■

Proof of Proposition 5: Differentiating both equations of (20), we have $\dot{w} = \dot{m}$,

$$\frac{\partial q(p,\ t)}{\partial t} + \frac{\partial q(p,\ t)}{\partial p} 2\dot{m} + m\left[\frac{\partial^2 q(p,\ t)}{\partial p^2} 2\dot{m} + \frac{\partial^2 q(p,\ t)}{\partial p \partial t}\right] + \dot{m}\frac{\partial q(p,\ t)}{\partial p} = 0$$

and thus

$$\dot{m}\left[3\frac{\partial q(p,\ t)}{\partial p} + 2m\frac{\partial^2 q(p,\ t)}{\partial p^2}\right] = -\frac{\partial q(p,\ t)}{\partial t} - m\frac{\partial^2 q(p,\ t)}{\partial p \partial t}.$$

Taking into account (21) and $\dot{w} = \dot{m}$, we observe monotonous evolution similar to that obtained for centralized pricing, but with respect to the wholesale price and the retailer's margin. ■