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Multi-stage newsboy problem: A dynamic model

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Abstract

The newsboy problem is a well-known operations research model. Its various extensions have been applied to managing capacity and evaluating advanced orders in manufacturing, retail and service industries.

This paper focuses on a dynamic, continuous-time generalization of the single-period newsboy problem. The problem is characterized by a number of the newsboys whose operations are organized and controlled in sequential stages. The objective is to minimize shortage and surplus costs occurring at the end of the period as in the classical newsboy problem, as well as intermediate surplus costs incurring at each time point along the period. We prove that this continuous-time problem can be reduced to a number of discrete-time problems. On this basis, a polynomial-time combinatorial algorithm is derived to find globally optimal solution when the system satisfies a certain capacity condition.

Keywords: Multi-stage production; Newsboy problem; Optimal control

1. Introduction

The classical, single-period newsboy problem is to find products order quantity that either maximizes the expected profit or minimizes the expected costs of overestimating and underestimating probabilistic demand. The newsboy problem has attracted considerable attention since the pioneering papers of Arrow et al. (1951), and Morse and Kimbal (1951). An extensive literature review on various extensions of the classical newsboy problem and related multi-stage, inventory control models can be found, for example, in Khouja (1999) and Silver et al. (1998). Among numerous extensions to this problem suggested so far one can find different objectives (see for example, Chung, 1990; Eeckhoudt et al., 1995), different supplier pricing policies (Kabak and Weinberg, 1972; Lin and Kroll, 1997), different newsvendor pricing policies and discount structures (Khouja, 1995; Lau and Lau, 1988), random yield of defective units (Henig and Gerchak, 1990) or of production capacity (Ciarolo et al., 1994),

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multi-products (Lau and Lau, 1996; Chang and Lin, 1991) and a number of subperiods to prepare for the selling season (Hausman and Peterson, 1972; Matsuo, 1990; Bitran et al., 1986). The idea behind the last type of extension is that there may be many periods to produce the items, which will be sold in a single season. Such dynamic models thus stress the importance of timing in producing or purchasing the items. These models commonly utilized special product (or product family) and demand parameters to optimize operations under limited production capacity over each subperiod (Bitran et al., 1986; Hausman and Peterson, 1972). The former work, which dealt with several families of style goods, resulted in a stochastic mixed-integer programming problem, the latter formulated the single and multi-product cases as dynamic programming problems. Both studies allow for forecast updates, but suggest heuristic methods to provide an approximate solution. Matsuo (1990) observed that a limitation of these works is that they include discrete production subperiods to assign production and suggested a continuous-time heuristic approach for improving the objective function value when approximating the optimal solution to the problem.

In this paper, we consider a multi-stage, continuous-time extension to the classical single-period newsboy problem. Products flow from one stage to the next. We assume we do not know the demand during the planning horizon, but we do know the *cumulative* demand at the end of the planing horizon. (This is the same assumption made in classical newsboy problems.) Forecast updates considered in the aforementioned newsboy extensions by Bitran et al. (1986) and Matsuo (1990) are not available along the horizon. The objective is to adjust the production rates during the planning horizon in order to minimize total costs. In this paper the total costs include shortage or surplus costs occurring at the end of the planning horizon for the last stage (as considered in the classical newsboy problem), as well as the surplus costs at the remaining stages during the planning horizon. Note, that in contrast to the classical model, the dynamic continuous-time approach enables us to make a decision at each point of time. Thus it takes into account all associated costs during the planning horizon.

A typical example of the dynamic system described above is found in a commercial yogurt production which is a short run, continuous manufacturing process. The process is composed of the following controllable stages: pretreatment of milk (standardization, fortification, lactose hydrolysis), homogenization, heat treatment, cooling to incubation temperature, inoculation with starter, fermentation, cooling, post-fermentation treatment (flavoring, fruit addition, pasteurization), refrigeration/freezing, and packaging. The process suffers from high demand variability as well as extremely tight flow-time performance requirements. Therefore, scheduling and controlling the key tasks for the incoming demand forecasts is of great importance to avoid overproduction, which leads to product spoilage, and underproduction, which results in the loss of sales.

We use the maximum principle to study optimal behavior of this dynamic system. As a result, the continuous-time, multi-stage newsboy problem is reduced to a number of discrete problems of sorting newsboys, determining optimal order quantity, timing and allocating operation rates over the planning horizon. The advantage of this approach is that it enables us to develop a polynomial time, combinatorial algorithm which provides globally optimal solution to the problem if the system has sufficient capacity.

Note, that the model can be further generalized to encompass multiple production lines utilizing common intermediate products and forecasts for different end items as well as forecast updates. Specifically, as the model is in continuous time, any forecast update points can be represented in the objective function and, therefore, the approach suggested in the paper can be applied. An optimal solution (which we consider an important result of this paper) is unlikely to be possible to find in such a case. However, plausible heuristics might be developed, which is a direction for our future research.

Since the multi-stage, continuous-time newsboy problem can be straightforwardly applied to production flow control of serial, one-product-type manufacturing systems, we further introduce the problem in the context of a flow shop.

2. Problem formulation

Consider a tandem manufacturing system containing I machines depicted in Fig. 1. It produces a single product type to satisfy a cumulative demand D for the product type by the end of a planning horizon, T. This system can be described by the following differential equations:

$$\begin{aligned} &\dot{X}_i(t) = u_i(t) - u_{i+1}(t), \quad X_i(0) = 0, \quad i = 1,2,\ldots,I-1; \\ &\dot{X}_i(t) = u_i(t), \quad X_i(0) = 0, \;\; i = I, \end{aligned} \tag{1}$$

where $X_i(t)$ is the surplus level at the buffer located after the ith machine (denoted m_i); $u_i(t)$ is the production rate of m_i. In this paper, $u_i(t)$ is the control variable the value of which can be instantly set within certain bounds:

$$0 \leqslant u_i(t) \leqslant U_i, \quad i = 1,2,\ldots,I, \tag{2}$$

with U_i being the maximal production rate of machine i. The product demand D is a random variable representing yield amount of the product type and characterized by probability density $\varphi(D)$ and cumulative distribution $\Phi(a) = \int_0^a \varphi_d(D)\,\mathrm{d}D$ functions respectively. For each planning horizon T, there will be a single realization of D which is known only by time T. Therefore, the decision has to be made under these uncertain conditions before the production starts.

Eq. (1) present the flow of products through a buffer placed between two consecutive machines. If the buffer is intermediate, this flow is determined at each point of time by the difference between the current production rates of the two consecutive machines. If the buffer is after the last machine and is intended for the finished products, then the flow is determined by the production rate of the last machine. The products are accumulated in this buffer in order to be delivered to the customers at the end of the production horizon. The difference between the cumulative production and the cumulative demand, $X_I(T) - D$, is the surplus level of the last machine m_I. If the cumulative demand exceeds the cumulative production of m_I, i.e., the surplus is negative, a penalty will have to be paid for the lost sales. On the other hand, if $X_I(T) - D > 0$ overproduction cost is incurred at the end of the planning horizon. Furthermore, inventory costs are incurred when buffer levels of the machines are positive, $X_i(t) > 0$, $i = 1,2,\ldots,I$. Negative buffer levels are prohibited:

$$X_i(t) \geqslant 0 \quad i = 1,2,\ldots,I-1. \tag{3}$$

Note, that (1) implies that $X_I(t) \geqslant 0$ always holds.

The objective is to find such controls $u_i(t)$ that satisfy constraints (1)–(3) while minimizing the following expected cost over the planning horizon T:

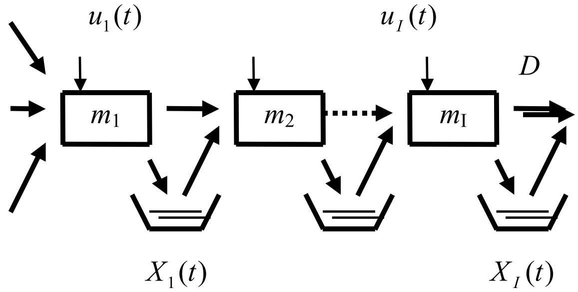

Fig. 1. Multi-stage tandem production system.

$$J = E\left[\int_0^T \sum_i C_i(X_i(t))\,dt + P(X_I(T) - D)\right] \to \min. \tag{4}$$

Linear and piece-wise linear cost functions are used for the inventory and surplus/backlog costs respectively,

$$C_i(X_i(t)) = c_i X_i(t), \tag{5}$$

$$P(Z) = p^+ Z^+ + p^- Z^-, \tag{6}$$

where $Z^+ = \max\{0, Z\}$ and $Z^- = \max\{0, -Z\}$.

3. Equivalent deterministic formulation

Let us substitute (5) and (6) into the objective (4). Then, given probability density $\varphi(D)$ of the demand, we find:

$$\begin{aligned} J &= \int_0^T \sum_i c_i X_i(t)\,dt + \int_0^\infty p^+ \max\{0, X_I(T) - D\}\varphi(D)\,dD + \int_0^\infty p^- \max\{0, D - X_I(T)\}\varphi(D)\,dD \\ &= \int_0^T \sum_i c_i X_i(t)\,dt + \int_0^{X_I(T)} p^+ (X_I(T) - D)\varphi(D)\,dD + \int_{X_I(T)}^\infty p^- (D - X_I(T))\varphi(D)\,dD. \end{aligned} \tag{7}$$

The new objective (7) is subject to constraints (1)–(3) which together constitute a deterministic problem equivalent to the stochastic problem (1)–(6).

Lemma 1. *Problem* (1)–(3) *and* (7) *is unimodal.*

Proof. Since constraints (1)–(3) are linear, cost functions $C_i(X_i(t))$ are linear and the sum of convex functions is a convex function, the proof is straightforwardly obtained by verifying whether the second term of objective function (7)

$$R = \int_0^{X_I(T)} p^+ (X_I(T) - D)\varphi(D)\,dD + \int_{X_I(T)}^\infty p^- (D - X_I(T))\varphi(D)\,dD$$

is convex with respect to $X_I(T)$:

$$\frac{\partial^2 R}{\partial X(T)^2} = (p^+ + p^-)\frac{\partial \Phi(X_I(T))}{\partial X_I(T)} \geqslant 0. \qquad \square$$

4. Dual formulation and optimality conditions

To study the equivalent deterministic problem we formulate a dual problem with costate variables $\psi_i(t)$ satisfying the following dual equations:

$$d\psi_i(t) = c_i\,dt - d\mu_i(t), \quad i = 1, 2, \ldots, I - 1 \quad \text{and} \quad \dot{\psi}_I(t) = c_I \tag{8}$$

with transversality (boundary) constraints:

$$\psi_i(T + 0) = 0, \quad i = 1, 2, \ldots, I - 1;$$

$$\psi_I(T) = -\frac{\partial\left[\int_0^{X_I(T)} p^+(X_I(T)-D)\varphi(D)\,\mathrm{d}D + \int_{X_I(T)}^{\infty} p^-(D-X_I(T))\varphi(D)\,\mathrm{d}D\right]}{\partial X_I(T)}$$
$$= \int_0^{X_I(T)} p^+\varphi(D)\,\mathrm{d}D + \int_{X_I(T)}^{\infty} p^-\varphi(D)\,\mathrm{d}D,$$

that is

$$\psi_I(T) = -p^+\Phi(X_I(T)) + p^-(1-\Phi(X_I(T))). \tag{9}$$

Left-continuous functions of bounded variation, $\mu_i(t)$, are due to the state constraint (3) and present possible jumps of the corresponding costate variables when $X_i(t)=0$. These jumps satisfy the non-negativity

$$\mathrm{d}\mu_i(t) \geqslant 0 \tag{10}$$

and complementary slackness condition

$$\int_0^T X_i(t)\,\mathrm{d}\mu_i(t) = 0. \tag{11}$$

The Hamiltonian is the objective for the dual problem, which is maximized by the optimal controls according to the maximum principle (Maimon et al., 1998):

$$H = -\sum_i c_i X_i(t) + \sum_{i\neq I}\psi_i(t)(u_i(t)-u_{i+1}(t)) + \psi_I(t)u_I(t) \to \max. \tag{12}$$

By rearranging only control dependent terms of the Hamiltonian we obtain:

$$H(t) = \sum_{i>1} u_i(t)(\psi_i(t)-\psi_{i-1}(t)) + u_1(t)\psi_1(t).$$

Since this term is linear in $u_i(t)$, it can be easily verified that the optimal production rate that maximizes $H(t)$ is

$$u_i(t) = \begin{cases} U_i, & \text{if } \psi_i(t)-\psi_{i-1}(t)>0,\ \forall i>1 \text{ and } \psi_i(t)>0,\ i=1 \text{ (production regime–PR)};\\ w\in[0,U_i], & \text{if } \psi_i(t)-\psi_{i-1}(t)=0,\ \forall i>1 \text{ and } \psi_i(t)=0,\ i=1 \text{ (singular regime–SR)};\\ 0, & \text{if } \psi_i(t)-\psi_{i-1}(t)<0,\ \forall i>1 \text{ and } \psi_i(t)<0,\ i=1 \text{ (idle regime–IR)}. \end{cases} \tag{13}$$

Thus under the optimal control the ith machine m_i can either be idle (denoted $m_i \in \mathrm{IR}$), working with its maximal production rate ($m_i \in \mathrm{PR}$), or entering the singular regime ($m_i \in \mathrm{SR}$). Since the primal problem is unimodal (see Lemma 1), the maximum principle provides not only the necessarily, but also the sufficient conditions of optimality. Therefore, all triplets $(u_i(t), X_i(t), \psi_i(t))$ that satisfy the primal (1)–(3), the dual (8)–(13) will minimize the objective function (7).

We next study the singular regime as its underlying controls are not uniquely determined in optimality conditions (13). To ensure the uniqueness of the solutions over this regime, we need the following assumption:

Assumption 1. $c_i \neq c_{i+1}$ for all $1 \leqslant i < I$.

Lemma 2. *If* $m_i \in \mathrm{SR}$ *in a time interval* τ *then* $X_{i-1}(t)=0$ *and/or* $X_i(t)=0$.

Proof. By definition, in SR $\psi_i(t) = \psi_{i-1}(t)$, $t \in \tau$. Differentiating this equality we have:

$$d\psi_i(t) = d\psi_{i-1}(t), \quad i > 1, \tag{14}$$

$$d\psi_i(t) = 0, \quad i = 1. \tag{15}$$

By substituting the corresponding costate equations, we then find:

$$c_i\,dt - d\mu_i(t) = c_{i-1}\,dt - d\mu_{i-1}(t), \quad i > I \quad \text{and} \quad c_i\,dt - d\mu_i(t) = 0, \ i = 1.$$

By taking into account (10) and (11), we conclude that the last equalities can be satisfied if and only if $X_{i-1}(t) = 0$ and or $X_i(t) = 0$ for $i \neq I$. □

From the system equation (1) we observe:

Fact 1. *If in a time interval* τ, $X_i(t) = 0$, $1 \leqslant i < I - 1$ *then* $u_i(t) = u_{i+1}(t)$, *and if* $i = I$, $u_i(t) = 0$.

5. Optimal solution

To study extremal behavior of the system, we need to distinguish between two types of machines and buffers.

Definition 1. Machine i', is a restricting machine if either $i' = I$ or $U_i' < U_i$, for all $I \geqslant i > i'$, $i' \neq I$.

Notice, according to the definition, the restricting machine that is closest to the first machine is the bottleneck machine, which has the smallest maximum production rate.

Definition 2. Buffer placed after machine i', is a restricting buffer if either $i' = I$ or $c_{i'} < c_i$, for all $I \geqslant i > i'$, $i' \neq I$.

Intuitively, one may guess that the most important question is to derive optimal behavior of the restricting machines while behavior of the non-restricting machines followed by non-restrictive buffers is completely determined by the restricting machines. According to Definitions 1 and 2, inventory costs and machine maximum production rates determine whether the corresponding machines and/or buffers are restricting. These notions generalize the well-known notion of the bottleneck and provide an important insight on ranking machines in terms of their optimal control. Based on this insight, in what follows, we assume that the production system consists only of restricting machines. Once optimal solution for such a system is found, we then generalize it for all types of machines. Moreover, in order for the problem to be tractable we need the following assumption.

Assumption 2. If machine i', is a non-restricting machine, then its buffer is also non-restricting.

We now use a constructive approach to solve the problem. That is, we first propose a solution, and then we show this solution is indeed optimal. To formalize the solution we denote by J the total number of restricting buffers and by $R(j)$, $j = 1, 2, \ldots, J$ their indexes.

The optimal control policy we are proposing for each subset $S_j = \{m_{R(j)}, m_{R(j)-1}, \ldots, m_{R(j-1)+1}\}$, $R(0) = 0$, $j = 1, 2, \ldots, J$ of the restricting machines is the following.

- Use the IR–PR (no production and then production at the maximum rate of $m_{R(j-1)+1}$) production sequence for $m_{R(j-1)+1}$ and IR–SR (no production and then singular production at the maximum rate of $m_{R(j-1)+1}$ with the same switching time t_j for each m_i, $i = R(j), R(j) - 1, \ldots, R(j-1) + 2$.

This policy is more rigorously defined in the following.

Policy A. *Consider a system with I restricting machines, J restricting buffers and J switching points,* $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_J \leqslant T$. *The behavior we are proposing for each subset of machines* $S_j = \{m_{R(j)}, m_{R(j)-1}, \ldots, m_{R(j-1)+1}\}$, $R(0) = 0$, $j = 1, 2, \ldots, J$ *is*

(i) $u_i(t) = 0$ *for* $0 \leqslant t < t_j$, $u_i(t) = U_{R(j-1)+1}$ *for* $t_j \leqslant t \leqslant T$, $i = R(j), R(j) - 1, \ldots, R(j-1) + 1$.
(ii) $X_i(t) = 0$ *for* $0 \leqslant t \leqslant T$, $i = R(j) - 1, \ldots, R(j-1) + 1$; $X_{R(j)}(t) = 0$ *for* $0 \leqslant t \leqslant t_j$, $X_{R(j)}(t) > 0$ *for* $t_j < t < T$, $X_{R(j)}(T) = 0$ *for* $R(j) \neq I$.
(iii) $\psi_i(t) = \psi_{R(j-1)}(t)$, *for* $0 \leqslant t \leqslant t_j$, $\dot{\psi}_i(t) = c_{R(j)}$, *for* $t_j \leqslant t \leqslant T$, $\psi_0(t) = \psi_1(t)$, $i = R(j), R(j-1), \ldots, R(j-1) + 1$, $j > 1$; $\psi_i(t) = \psi_{R(1)}(t)$, $\psi_i(t_1) = 0$, $\dot{\psi}_i(t) = c_{R(1)}$ *for* $0 \leqslant t \leqslant T$, $\psi_i(t_1) = 0$, $i = R(1), R(1) - 1, \ldots, 1, j = 1$.

We now show the proposed behavior for restricting machines satisfies the dual system (8)–(11) and the maximum principle based optimality conditions (13).

Lemma 3. *If all machines are restricting, then Policy* A *provides the optimal solution.*

Proof. First note that according to Policy A, $X_i(T) = 0$ for $i = 1, 2, \ldots, I - 1$, which with respect to (10) and (11) implies that the transversality constraints $\psi_i(T + 0) = 0$ are satisfied with instant jumps $d\mu_i(T)$.

Consider the first subset of machines, $S_1 = \{m_{R(1)}, m_{R(1)-1}, \ldots, m_1\}$. According to (i) of Policy A $u_i(t) = 0$ for $0 \leqslant t < t_1$, $u_i(t) = U_1$ for $t_1 \leqslant t \leqslant T$, $i = R(1), R(1) - 1, \ldots, 1$. This control is feasible since the production system consists of only restricting machines, that is $U_1 < U_i$.

Next, according to (ii) of Policy A, $X_i(t) = 0$ for $0 \leqslant t \leqslant T$, $i = R(1) - 1, \ldots, 1$; $X_{R(1)}(t) = 0$ for $0 \leqslant t \leqslant t_1$, $X_{R(1)}(t) > 0$ for $t_1 < t < T$, $X_{R(1)}(T) = 0$, which evidently satisfies the state equations (1) if $u_{R(1)+1}(t) = 0$ for $0 \leqslant t < t_2$, $u_{R(1)+1}(t) = U_{R(1)+1}$ for $t_2 \leqslant t \leqslant T$, as stated in Policy A(ii) and

$$t_1 U_1 = t_2 U_{R(1)+1}. \tag{16}$$

Consider now solution for the costate variables. According to Policy A(iii), $\psi_i(t) = \psi_{R(1)}(t)$, $\psi_i(t_1) = 0$ and $\dot{\psi}_i(t) = c_{R(1)}$ for $0 \leqslant t \leqslant T$, $i = R(1), R(1) - 1, \ldots, 1$, $j = 1$, which meets costate equations (8) for the determined behavior of the state variables. This also implies that $\psi_i(t) < 0$ (idle regime, $u_i(t) = 0$ according to (13)) for $0 \leqslant t < t_1$, and $\psi_i(t) > 0$ (full production $u_i(t) = U_1$ according to (13)) for $t_1 \leqslant t \leqslant T$, $i = 1$. Furthermore, condition $\psi_i(t) = \psi_{R(1)}(t)$ satisfies the singular regime from (13), Lemma 2 conditions and Fact 1 for $i = R(1), R(1) - 1, \ldots, 1$. Thus (1), (8)–(11) are satisfied and (12) is maximized.

Similarly, by considering subsequent subsets of machines, $S_j = \{m_{R(j)}, m_{R(j)-1}, \ldots, m_{R(j-1)+1}\}$, $j = 2, \ldots, J$, one can verify that the proposed solution satisfies the state and costate equations and the optimality conditions (13) if the switching times are set as

$$t_j U_{R(j-1)+1} = t_{j+1} U_{R(j)+1}, \quad j < J. \tag{17}$$

The only difference is that the last machine $I = R(J)$ does not have a predecessor and its switching point, $t_J = X_I(T)/U_{R(J-1)+1}$ is determined so that $X_I(T)$ satisfies the corresponding transversality condition (9). □

Based on Policy A we can solve the two-point boundary value problem (1), (2), (8)–(11) to find the switching time points.

We first note that Policy A(ii) condition $X_i(T) = 0$ for $i = 1, 2, \ldots, I - 1$ along with (17) implies:

$$t_j = T - \frac{X_I(T)}{U_{R(j-1)+1}} \quad \text{for } j = 1, 2, \ldots, J. \tag{18}$$

Then, by integrating the costate equations (8) with boundary conditions from Policy A(iii), we find:

$$\psi_I(T) = \sum_{1<j\leqslant J} c_{R(j-1)}(t_j - t_{j-1}) + c_I(T - t_J). \tag{19}$$

Finally, by substituting (18) into (19) and taking into account the corresponding transversality condition from (9), we obtain the following equation in unknown $X_I(T) = 0$:

$$\sum_{1<j\leqslant J} c_{R(j-1)}\left(\frac{1}{U_{R(j-2)+1}} - \frac{1}{U_{R(j-1)+1}}\right) + c_I \frac{1}{U_{R(J-1)+1}} = \frac{-p^+\Phi(X_I(T)) + p^-(1-\Phi(X_I(T)))}{X_I(T)}. \tag{20}$$

Eq. (20) allows determining optimal production or order quantity $X_I(T)$ for the multi-stage, continuous-time manufacturing system. Note, that by setting all inventory costs at zero in Eq. (20), one can now obtain well know in the operations research literature economic order quantity with limited sales period for the classical, single-stage newsboy problem as stated by Fact 2.

Fact 2. *If $c_i = 0$ for $i = 1,\ldots,I$, then $\Phi(X_I(T)) = p^-/(p^+ + p^-)$.*

We now study the effect of non-restricting machines attended by non-restricting buffers on the optimal behavior of the system. The approach is similar to that for the restricting machines. We denote by K the total number of the restricting machines and by $Q(k)$, $k = 1,2,\ldots,K$ their indexes. Next, we propose an optimal control policy for each subset $S_k = \{m_{Q(k)-1}, m_{Q(k)-2},\ldots,m_{Q(k-1)+1}\}$, $Q(0) = 0$, $k = 1,2,\ldots,K$ of the non-restricting machines as follows:

- If $k > 1$, use SR (the singular production at the rate of adjacent upstream restricting machine $m_{Q(k-1)}$) for each m_i, $i = Q(k)-1, Q(k)-2,\ldots,Q(k-1)+1$.
- If $k = 1$, use SR (the singular production at the rate of adjacent downstream restricting machine $m_{Q(k)}$) for each m_i, $iQ(k)-1, Q(k)-2,\ldots,Q(k-1)+1$.

This policy is more rigorously defined as follows.

Policy B. *Consider a system with I restricting machines, J restricting buffers and J switching points, $0\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_J\leqslant T$. The behavior we are proposing for each subset of non-restricting machines $S_k = \{m_{Q(k)-1}, m_{Q(k)-2},\ldots,m_{Q(k-1)+1}\}$, $Q(0) = 0, k = 1,2,\ldots,K$ is*

(i) *if* $k > 1$, $u_i(t) = u_{Q(k-1)}(t)$, *otherwise* $u_i(t) = u_{Q(k)}(t)$ *for* $0\leqslant t\leqslant T$, $i = Q(k)-1, Q(k-2),\ldots, Q(k-1)+1$;
(ii) $X_i(t) = 0$ *for* $0\leqslant t\leqslant T$, $i = Q(k)-1, Q(k)-2,\ldots,Q(k-1)+1$;
(iii) *if* $k > 1$, $\psi_i(t) = \psi_{Q(k-1)}(t)$, *otherwise* $\psi_i(t) = 0$ *for* $0\leqslant t\leqslant T$, $i = Q(k)-1, Q(k-2),\ldots,Q(k-1)+1$.

We now show the proposed behavior for non-restricting machines satisfies the dual system (8)–(11), the maximum principle based optimality conditions (13), and does not effect the optimal behavior of the restricting machines determined by Policy A.

Lemma 4. *If all restricting machines satisfy Policy* A *and all non-restricting machines satisfy Policy* B, *then these policies provide the optimal solution.*

Proof. The proof is straightforward. The policy described in Policy B satisfies Lemma 2, that is the dual system (8)–(11), and conditions (13) for the non-restricting machines. Moreover, one can readily observe, that since the costate variables of non-restricting machines for $k > 1$ simply copy those for the adjacent upstream restricting machines and set at zero otherwise, the optimality conditions (13) for the restricting machines do not change. Finally, copying for $k > 1$, $\psi_i(t) = \psi_{Q(k-1)}(t)$ is feasible because of two facts: Assumption 2, which implies $c_i \geqslant c_{q(k-1)}$ and $X_i(t) = 0$ for $0 \leqslant t \leqslant T$, which implies any jumps $d\mu_i(t) \geqslant 0$ of the costate variables are allowed. □

6. Algorithm

The algorithm is straightforward and immediately follows from Policies A and B as described below.

INPUT: I; c_i, U_i, $i = 1, \dots, I$; p^+, p^-.

Step 1. Use Definitions 1 and 2 to determine and number restricting machines $Q(k)$, $k = 1, \dots, K$ and restricting buffers $R(j)$, $j = 1, \dots, J$.

Step 2. Consider only restricting machines. Use (20) to calculate $X_I(T)$. Form subsets $S_j = \{m_{R(j)}, m_{R(j)-1}, \dots, m_{R(j-1)+1}\}$, $R(0) = 0$, $j = 1, 2, \dots, J$.

Step 3. Use Eq. (18) to calculate J switching points. Use Policy A to set the optimal solution for all subsets of the restricting machines.

Step 4. Consider all machines. Form subsets of the non-restricting machines $S_k = \{m_{Q(k)-1}, m_{Q(k)-2}, \dots, m_{Q(k-1)+1}\}$, $Q(0) = 0$, $k = 1, 2, \dots, K$.

Step 5. Use Policy B to set the optimal solution for all subsets of the non-restricting machines.

OUTPUT: Optimal controls $u_i(t)$ for $0 \leqslant t \leqslant T$, $i = 1, 2, \dots, I$.

Lemma 5. *Given $X_I(T)$ which satisfies Eq.* (20), *if $X_I(T)/U_{Q(1)} \leqslant T$, then problem* (1)–(6) *is solvable in $O(I)$ time.*

Proof. According to Lemmas 2–4, the solution presented by Policies A and B is optimal, if the switching points determined by (18) are feasible, i.e., there is enough capacity to produce $X_I(T)$ by the end of the planning horizon. The feasibility is straightforwardly provided by requiring $t_1 \geqslant 0$. This inequality, by taking into account Eq. (18) and the fact that the first restricting machine $R(0) + 1$ is $Q(1)$, results into $X_I(T)/U_{Q(1)} \leqslant T$, stated in the lemma. Moreover, due to Lemma 1, this solution is globally optimal. Finally, provided a solution of the optimal order quantity equation (20), each step of the algorithm evidently requires only $O(I)$ operations. □

Remark. Lemma 5 estimates the complexity of solving problem (1)–(6) provided optimal order equation (20) can be resolved analytically. However, analytical solution is not always available. One can readily observe that Eq. (20) is monotone in the unknown $X_I(T)$, which implies that it can be easily solved numerically to any desired precision.

7. Example

Consider the uniform distribution,

$$\varphi(D) = \begin{cases} \frac{1}{d}, & \text{for } 0 \leqslant D \leqslant d; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\Phi(a) = a/d$ and Eq. (20) takes the following form:

$$X_I(T) = \frac{p^-}{\frac{(p^+ + p^-)}{d} + \sum_{1<j\leqslant J} c_{R(j-1)} \left(\frac{1}{U_{R(j-2)+1}} - \frac{1}{U_{R(j-1)+1}} \right) + c_I \frac{1}{U_{R(J-1)+1}}}. \tag{21}$$

Table 1
Parameters of the five-machine system

Parameters	Machine 1	Machine 2	Machine 3	Machine 4	Machine 5
U_i	3	2	4	7	6
c_i	2.5	0.5	3	2	1

To illustrate each step of the algorithm, we consider a small, five-machine serial production system. The input data for such a system is presented in Table 1.

In addition $T = 5$ time units, $d = 24$ product units, $p^+ = 1\$$ per product unit and $p^- = 2\$$ per product unit.

Step 1 identifies sets of three restricting machines ($K = 3$) as $Q(1) = 2$, $Q(2) = 3$, $Q(3) = 5$ and two restricting buffers ($J = 2$) as $R(1) = 2$, $R(2) = 5$, respectively. Considering only restricting machines 2, 3 and 5 at Step 2 results in

$$X_5(T) = \frac{p^-}{\frac{(p^+ + p^-)}{d} + c_2\left(\frac{1}{U_2} - \frac{1}{U_3}\right) + c_5\frac{1}{U_5}} = 4.8 \text{ product units,}$$

$$S_1 = \{m_2\} \quad \text{and} \quad S_2 = \{m_5, m_3\}.$$

Two switching points are calculated at Step 3, as $t_1 = T - (X_5(T)/U_2) = 2.6$ time units and $t_2 = T - (X_5(T)/U_3) = 3.8$ time units. Then, based on Policy A, the optimal solution is set for restricting machines:

$$u_2(t) = 0 \quad \text{for } 0 \leqslant t < 2.6 \quad \text{and} \quad u_2(t) = 2 \quad \text{for } 2.6 \leqslant t \leqslant 5,$$

$$u_3(t) = u_5(t) = 0 \quad \text{for } 0 \leqslant t < 3.8 \quad \text{and} \quad u_3(t) = u_5(t) = 4 \quad \text{for } 3.8 \leqslant t \leqslant 5.$$

Next, Step 5 forms subsets of non-restricting machines as $S_1 = \{m_1\}$, $S_2 = \{\varnothing\}$ and $S_3 = \{m_4\}$. Finally, according to Policy B, the optimal solution for the non-restricting machines is set at Step 5:

$$u_1(t) = u_2(t) \quad \text{and} \quad u_4(t) = u_3(t) \quad \text{for } 0 \leqslant t \leqslant 5.$$

8. Conclusion

A dynamic, continuous-time extension of the well-known single-period newsboy problem is introduced to incorporate such important factors as time, multiple stages and controllable operation rates. The problem is presented in the context of a serial manufacturing system consisting of tandem machines and buffers placed after them. With the aid of the maximum principle the continuous-time problem is reduced to determining optimal production order quantity, ranking machines and buffers, calculating a limited number of switching time points and assigning production rates over the switching points with respect to the machine and buffer ranks. We show that if the system has sufficient capacity and the equation derived for the optimal production order can be solved analytically, the original problem is solvable in $O(I)$ time, where I is the number of stages. Furthermore, even if the optimal order equation is not solvable analytically, it is always solvable in polynomial time numerically to any required precision.

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