From the definition of $\mathcal{Q}(\cdot)$, one can prove through lengthy but simple calculations that

$$\frac{\left|\frac{\partial \mathcal{Q}}{\partial x}(x,t)\right|^2}{1+\mathcal{Q}^2(x,t)} \le \left(16|Q|^2 + 16\rho^2|Q|^2 + 8S(\lambda_0)^2\right)|x|^2. \quad (55)$$

We deduce that (39) holds with $\varepsilon = 1/(128(|Q|^2 + \rho^2|Q|^2 + S(\lambda_0)^2))$.

2) Now, we prove (40), which is satisfied if, for a given new constant $c \ge 0$, there exist λ_0 and ρ such that

$$c\rho\lambda_0^2 \left|x^\top Q\right| \sigma(|x|_Q)^2 \left|D(t)^\top Qx\right|^2 \le |x|^2 + \rho\lambda_0\sigma(|x|_Q)|x|_Q \left|D(t)^\top Qx\right|^2 \quad (56)$$

$$c|x^\top Q_{\lambda_0}(t)|\lambda_0^2 \frac{\sigma(|x|_Q)^2}{|x|_Q^2} \left|D(t)^\top Qx\right|^2 \le |x|^2 + \rho\lambda_0\sigma(|x|_Q)|x|_Q \left|D(t)^\top Qx\right|^2. \quad (57)$$

Using $|\sigma(\cdot)| \le 1$, one can check readily that (56) holds if $c\lambda_0|x^\top Q| \le |x|_Q$. This inequality is satisfied if λ_0 is sufficiently small.

Inequality (57) is satisfied if, with λ_0 chosen such that (57) holds, there exists $\rho \ge 0$ such that

$$c\lambda_0^2 S(\lambda_0)|x| \frac{\sigma(|x|_Q)^2}{|x|_Q^2} \left|D(t)^\top Qx\right|^2 \le |x|^2 + \lambda_0\rho\sigma(|x|_Q)|x|_Q \left|D(t)^\top Qx\right|^2. \quad (58)$$

Using the triangular inequality, we deduce that this inequality is satisfied if

$$c\lambda_0^3\sigma(|x|_Q)^3 \left|D(t)^\top Qx\right|^2 \le \rho|x|_Q^5 \quad (59)$$

where $c > 0$ is a constant independent of ρ and λ_0. Since $|D(t)|$ is smaller than a strictly positive constant for all $t \ge 0$, we deduce that one can choose a sufficiently large ρ such that (59) is satisfied. △

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Optimal Control of a Resource-Sharing Multiprocessor With Periodic Maintenance

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***Abstract*—Shared resources and the processes that control them play a critical role in the functioning of concurrent systems. This note analyzes the production control of a workstation producing a number of products concurrently. The workstation is periodically stopped for maintenance. The objective of the production control is to minimize inventory and backlog costs over an infinite time horizon. Using the maximum principle and under the so-called agreeable cost structure, we derive the optimal production control. We prove that under this cost structure, the problem can be solved in polynomial time.**

I. INTRODUCTION

Sharing resources is common in industrial applications. Advances in information technology have challenged Internet and database suppliers with the problem of providing a high level of service in the face of permanently growing demands. Specifically, the explosive growth of the Internet and the World Wide Web has brought a dramatic increase in the number of users that compete for the shared resources of distributed system environments [10]. Similarly, efficient control of shared resources is crucial for data base processing where online memory is allocated to each microprocessor [11], [8], as well as for designing a high-performance robot controller with multiple arithmetic processing units (APUs) [1]. Besides these modern applications, the classical problem of optimal scheduling of flexible-manufacturing systems, which comprise a number of work-cells where production resources are shared remains of significant practical importance [14].

As technology progresses, systems with shared resources become more complex. In order to realize the full economic life cycle of these systems, as well as to obtain maximum availability and reliability

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of such equipment, periodic maintenance is vital. The literature presents several methodologies for incorporating maintenance and system's control policies in stochastic environments (e.g., [2]–[4]) or in deterministic environments (e.g., [7] and [12]). Deterministic works typically proceed in two research directions. One is to suggest an efficient analytical production and/or maintenance policy. The main disadvantage is that strong limitations are introduced into the model in order to obtain a tractable solution. Among the limitations is the *a priori* assumption that the maximum maintenance rate is equal to a constant coefficient times the maximum production rate [6], application for a group of i.i.d. machines (e.g., [15]) or referring to only one product type [9]. The last work [9] studied a simple production system consisting of single machine and producing one part type. Demand rate was assumed to be constant and maintenance was performed on a periodic basis. Pontryagin's maximum principle was used to solve analytically the special case of one up–down cycle. Since analytic solutions are rarely available, the other direction involves using dynamic programming and the maximum principle for numerical approximation of the optimal control and preventive maintenance over fixed time points (e.g., [5] and [12]).

This note follows the analytical avenue in providing an optimal solution. To the best of our knowledge, besides the aforementioned work [9], there is no suggestion in the literature regarding an analytical solution for optimal scheduling of concurrent processing under periodically maintained shared resources. We extend the work of [9] by focusing on a deterministic production system consisting of one workstation, which produces a number of product-types (multiprocessor with shared resource). The workstation is periodically stopped for maintenance.

II. Statement of the Problem

Consider a workstation producing N product-types to satisfy demand rate d_n, $n = 1, 2, \ldots, N$. The workstation is periodically stopped for maintenance. Define t_s the time at which the production period starts, t_f the end of the maintenance period, P the production duration, and M the maintenance duration. We then have

$$t_f = t_s + P + M. \tag{1}$$

Assuming the system has reached the steady-state, then the cyclic behavior of the system can be described by the following differential equations:

$$\begin{aligned} \dot{X}_n(t) &= A(t)u_n(t) - d_n \\ X_n(t_s) &= X_n(t_f), \qquad n = 1, 2, \ldots, N \end{aligned} \tag{2}$$

where $X_n(t)$ is the surplus of product n at time t, if $X_n(t) \geq 0$, and the backlog, if $X_n(t) < 0$. $u_n(t)$ is the production rate and $A(t)$ is a periodic maintenance function defined as

$$A(t) = \begin{cases} 1, & \text{if } t_s \leq t < t_s + P; \\ 0, & \text{if } t_s + P \leq t < t_s + P + M. \end{cases} \tag{3}$$

The production rate is a control variable, which is bounded by the maximum production rate U_n for product n

$$0 \leq u_n(t) \leq U_n. \tag{4}$$

Since the workstation can produce a number of products concurrently, the following constraint ensures the production not exceed the capacity:

$$\sum_n \frac{u_n(t)}{U_n} \leq 1. \tag{5}$$

In order to assure that the demand can be fulfilled at each production cycle, we also need that

$$\sum_n \frac{d_n}{U_n} \leq \frac{P}{P+M}. \tag{6}$$

The objective is to find an optimal cyclic behavior $(u_n(t), X_n(t))$ of the workstation that satisfies constraints (2), (4), and (5) while minimizing the following piecewise linear cost functional:

$$J = \int_{t_s}^{t_f} \sum_n \left[c_n^+ X_n^+(t) + c_n^- X_n^-(t)\right] dt \tag{7}$$

where

$$X_n^+(t) = \max\{X_n(t), 0\} \quad X_n^-(t) = \max\{-X_n(t), 0\} \tag{8}$$

c_n^+ and c_n^- are the unit costs of storage (inventory) and backlog of product-type n, respectively.

In this note, we assume relatively large backlog costs are assigned to products that cause large inventory costs and vice versa, formalized as follows.

Assumption 1: The inventory and backlog costs are agreeable, that is, if $c_n^+ U_n > c_{n'}^+ U_{n'}$, then $c_n^- U_n > c_{n'}^- U_{n'}$ and vice versa, for $n, n' \in \Omega$, where $\Omega = \{1, \ldots, N\}$.

Without loosing generality, we also assume that if $c_n^+ U_n > c_{n'}^+ U_{n'}$ then $n > n'$, and if $n \neq n'$ then $c_n^+ U_n \neq c_{n'}^+ U_{n'}$, $n, n' \in \Omega$, where $\Omega = \{1, \ldots, N\}$.

III. Dual Formulation

The maximum principle [12] is used in this section to construct a dual problem. The Hamiltonian is the objective function of the dual problem and is maximized at every point of time by the optimal controls $u_n(t)$.

Applying the maximum principle to problem (2)–(7), the Hamiltonian, denoted H, is formulated as follows:

$$H = -\sum_n \left[c_n^+ X_n^+(t) + c_n^- X_n^-(t)\right] + \sum_n \psi_n(t)(u_n(t) - d_n). \tag{9}$$

The co-state variables, $\psi_n(t)$, (see [12]) satisfy the following dual differential equation with corresponding periodicity (boundary) condition:

$$\begin{aligned} \dot{\psi}_n(t) &= \begin{cases} c_n^+, & \text{if } X_n(t) > 0 \\ -c_n^-, & \text{if } X_n(t) < 0 \\ a,\ a \in [-c_n^-, c_n^+], & \text{if } X_n(t) = 0 \end{cases} \\ \psi_n(t_s) &= \psi_n(t_f). \end{aligned} \tag{10}$$

To determine the optimal production rate $u_n(t)$ when $A(t) \neq 0$, we consider the following four possible regimes, which are defined according to $U_n \psi_n(t)$.

1) Full Production (FP) Regime : This regime appears if there is an n such that $U_n \psi_n(t) > 0$, and $U_n \psi_n(t) > U_{n'} \psi_{n'}(t)$, $\forall n' \neq n$, $n, n' \in \Omega$. In this regime, according to (9), to maximize the Hamiltonian we should have $u_n(t) = U_n$ and $u_{n'}(t) = 0, \forall n' \neq n$, $n, n' \in \Omega$.

2) No Production (NP) Regime: If $U_n \psi_n(t) < 0$, $\forall n \in \Omega$. In this regime to maximize the Hamiltonian, we should have $u_n(t) = 0$, $\forall n \in \Omega$.

3) Singular Production (S-SP) Regime: This regime appears if there is an $S \subset \Omega$, the rank of S [the rank of S is defined as the number of units in S and denoted $R(S)$] is greater than 1, and $U_n \psi_n(t) = U_{n'} \psi_{n'}(t) > 0$, $\forall n, n' \in S$, and $U_n \psi_n(t) > U_m \psi_m(t)$, $\forall n \in S$, $m \notin S$. In this regime there is a set of products S (the active set) for which the Hamiltonian gradients $U_n \psi_n(t) > 0$ are equal to each other and are greater than all the other gradients at an interval of time.

4) Singular Production (Z-SP) Regime: This regime appears if there is a $Z \subset \Omega$ such that $U_n \psi_n(t) = U_{n'} \psi_{n'}(t) = 0$, $\forall n, n' \in Z$, and $U_n \psi_n(t) > U_m \psi_m(t)$, $\forall n \in Z$, $m \notin Z$. In this regime there is a set of products Z (the active set) for which the Hamiltonian gradients $U_n \psi_n(t) = 0$ and are greater than all the other gradients in an interval of time.

IV. PROPERTIES OF THE OPTIMAL SOLUTION

The optimal production rates under the singular production regimes are discussed in the following three lemmas.

Lemma 1: If there is an $n \in \Omega$ such that $U_n\psi_n(t) > 0$, then

1) $\sum_{m\in\Omega}(u_m(t)/U_m) = 1$;
2) if $u_n(t) > 0$ then $U_n\psi_n(t) \geq U_{n'}\psi_{n'}(t)$ for all $n,\ n' \in \Omega$.

Proof: Since the optimal control maximizes the Hamiltonian (9), the first part of the lemma must hold otherwise we could increase $u_n(t)$ to enlarge the Hamiltonian. To prove the second part of the lemma, assume there is an n' such that $U_n\psi_n(t) < U_{n'}\psi_{n'}(t)$. Also assume the portion of production capacity allocated to part n is $u_n(t)/U_n = \alpha$. Then $\alpha U_n\psi_n(t) < \alpha U_{n'}\psi_{n'}(t)$ and if the same capacity were allocated to part n' instead of n, Hamiltonian H could be increased. However, this violates the optimality assumption. ■

Lemma 2: Let the S-SP regime with its active set S be in a time interval τ. Then, the following hold for $t \in \tau$:

1) $X_{n^*}(t) \neq 0$ and
$$u_{n^*}(t) = U_{n^*}\left(1 - \sum_{\substack{n\in S,\\ n\neq n^*}} \frac{d_n}{U_n}\right)$$
for $n^* = \min_{n\in S} n$;
2) $u_n(t) = d_n,\ X_n(t) = 0$ for all $n \neq n^*,\ n \in S$;
3) $u_n(t) = 0$ for all $n \notin S$.

Proof: According to the definition of the S-SP regime
$$U_n\psi_n(t) = U_{n'}\psi_{n'}(t) > 0,\ t\in\tau \quad \text{for all } n,\ n' \in S, \tag{12}$$
and
$$U_n\psi_n(t) > U_l\psi_l(t),\ t\in\tau \quad \text{for all } n\in S,\ l \notin S. \tag{13}$$
By differentiating (12), we obtain
$$U_n\dot{\psi}_n(t) = U_{n'}\dot{\psi}_{n'}(t),\ t\in\tau. \tag{14}$$
Considering Assumption 1 and the definition of $\dot{\psi}_n(t)$ shown in (10), (14) can be met in only the following two cases:

Case 1) $X_n(t) = 0$ for all $n \in S$;
Case 2) $X_{n^*}(t) \neq 0$, and $X_n(t) = 0$ for all $n \neq n^*,\ n \in S$ with $n^* = \min_{n\in S} n$ and
$$u_{n^*}(t) = U_{n^*}\left(1 - \sum_{\substack{n\in S,\\ n\neq n^*}} \frac{d_n}{U_n}\right). \tag{16}$$

If $X_n(t) = 0$ in a time interval for some $n \in S$, then differentiating $X_n(t) = 0$ and using (2), we obtain
$$u_n(t) = d_n. \tag{17}$$
However, from (6), we have
$$1 - \sum_{n\in\Omega}\frac{d_n}{U_n} \geq 0, \text{ thus } \frac{u_{n^*}(t)}{U_{n^*}} = 1 - \sum_{\substack{n\in S,\\ n\neq n^*}}\frac{d_n}{U_n} \geq \frac{d_{n^*}}{U_{n^*}}.$$
In Case 1), $u_{n*}(t) = d_{n*}$. Thus, the previous inequality implies that the Hamiltonian in Case 2) will be larger than the Hamiltonian in Case 1) and, therefore, Case 2) provides the optimal control. The maximization of the Hamiltonian also demands that $u_n(t) = 0$ for all $n \notin S$. From (17) we have $u_n(t) = d_n$, for all $n \neq n^*,\ n\in S$. ■

Lemma 3: Let the Z-SP regime with its active set Z be in a time interval τ. Then $u_n(t) = d_n, X_n(t) = 0$ for all $n \in Z$ and $u_n(t) = 0$ for all $n \notin Z,\ t\in\tau$.

Proof: Consider the Z-SP regime which by definition satisfies
$$\psi_n(t) = 0,\ t\in\tau \quad \text{for all } n \in Z, \tag{18}$$
and
$$\psi_n(t) < 0,\ t\in\tau \quad \text{for all } n \notin Z.$$
First, if $\psi_n(t) < 0$ to maximize the Hamiltonian we must have $u_n(t) = 0$. Next, by differentiating (18), we obtain
$$\dot{\psi}_n(t) = \dot{\psi}_{n'}(t) = 0,\ t\in\tau \quad \text{for all } n,\ n' \in Z. \tag{19}$$
Using the same argument as in Lemma 2, we have
$$X_n(t) = 0,\ u_n(t) = d_n,\ t\in\tau \quad \text{for all } n\in Z. \tag{20}$$
■

The next lemma shows that there must be a Z-SP regime with its active set $Z = \Omega$ in some time interval τ.

Lemma 4: Let $\sum_n(d_n/U_n) < P/(P+M)$, then during the production period P there must be a Z-SP regime with its active set $Z = \Omega$ in some time interval τ.

Proof: We first notice that under the S-SP, Z-SP, and FP regimes $\sum_n(u_n(t)/U_n) = 1$. Also, based on the assumption of this lemma, we have $\sum_n(d_n/U_n)(P+M) < P$. Therefore, during the production duration P, if we only use the S-SP, Z-SP, and FP regimes, we would have $\sum_n(d_n/U_n)(P+M) < \sum_n(u_n(t)/U_n)P$, which implies the production would exceed demand. This violates our cyclic production assumption. Therefore, there must be a time period $\mathrm{P}_1 \subset \mathrm{P}$, during which $\sum_n(u_n(t)/U_n) < 1$, and the only possible regimes during P_1 are Z-SP and NP. If $Z \neq \Omega$, either Z-SP or NP will result in some product(s) being not produced. That is, there exists some n such that $u_n(t) = 0,\ t\in \mathrm{P}_1$. We now argue that this cannot be the optimal solution.

For such n that $u_n(t) = 0, t\in\mathrm{P}_1$, we must have $\psi_n(t) < 0$ under Z-SP or NP regimes. If $X_n(t) < 0$, then $\dot{\psi}_n(t) < 0$ and, thus, product n will not be produced again. This contradicts the cyclic production assumption. If $X_n(t) > 0$, then we can certainly reduce the overall cost by doing the following. We first reduce the production in the period before P_1 so that $X_n(t_1)0$, where t_1 is the starting time of P_1. We then let $u_n(t) = d_n, t\in\mathrm{P}_1$ maintain $X_n(t) = 0, t\in\mathrm{P}_1$. Both will reduce the inventory cost. Thus, we must have $u_n(t) \neq 0$, all $n\in\Omega, t\in\mathrm{P}_1$. Therefore, the only possible regime is Z-SP with $Z = \Omega$. ■

In the following, we will establish the optimal production sequence, starting from Z-SP regime with $Z = \Omega$. First, Lemma 5 shows that the regime following the aforementioned Z-SP regime must be an S-SP regime with $S = \Omega$.

Lemma 5: Let τ_1 and τ_2 be two consecutive time intervals, τ_2 following τ_1. If the Z-SP regime is in τ_1, then $u_n(t) > 0$ for all $n\in Z$, $t\in\tau_2$. Further, if $Z = \Omega$, then there is an S-SP regime in τ_2 with $S = \Omega$.

Proof: According to Lemma 3, $X_n(t) = 0$ and $\psi_n(t) = 0$ for $n\in Z,\ t\in\tau_1$. If $u_n(t)0,\ t\in\tau_2$, then from (2) and (10), we have $X_n(t) < 0,\ \dot{\psi}_n(t) < 0$, and $\psi_n(t) < 0,\ t\in\tau_2$. Therefore, $\psi_n(t) < 0$ for $t > t_1$, where t_1 is the starting time of τ_2 and product n will never be produced again. This contradicts the assumption of the cyclic production requirement. If $Z = \Omega$, then $u_n(t) > 0$ for all $n\in\Omega$, $t\in\tau_2$. This can only happen if S-SP regime is in τ_2 with $S = \Omega$. ■

We now state the relationship between two consecutive S-SP regimes.

Lemma 6: Let two S-SP regimes with their active sets S_1 and S_2 be in two consecutive time intervals τ_1 and τ_2, τ_2 following τ_1 and $m = \min_{n\in S_1} n$. If $X_m(t) > 0,\ t\in\tau_1$ and $m > n',\ \forall n'\in\Omega,\ n'\notin S_1$, then $S_1 = S_2 + m$.

Proof: If $n\in S_1,\ n > m$, then according to Lemma 2 we have $X_n(t) = 0,\ U_n\psi_n(t) = U_m\psi_m(t),\ t\in\tau_1$. If $u_n(t) = 0,\ t\in\tau_2$, then $X_n(t) < 0,\ t\in\tau_2$. Further, since $n > m$, if $X_n(t) < 0,\ U_n\dot{\psi}_n(t) < U_m\dot{\psi}_m(t)$ (see (10) and Assumption 1). Therefore, $U_n\psi_n(t) < U_m\psi_m(t)$ for all $t > t_1$, where t_1 is the starting time of τ_2. This ensures $u_n(t) = 0$ for all $t > t_1$ which contradicts the cyclic production requirement. Therefore, $u_n(t) > 0,\ t\in\tau_2$. Thus, $n\in S_2$.

We next show if $n\notin S_1$ then $n\notin S_2$. We first observe that by definition of an S-SP regime, $U_n\psi_n(t) > U_{n'}\psi_{n'}(t),\ \forall n\in S_1,\ n'\notin S_1$,

$t \in \tau_1$. Since $n > n'$ for $\forall\, n \in S_1,\, n' \notin S_1$ and $X_m(t) > 0,\, t \in \tau_1$ (assumptions of this lemma), we have $U_n\dot{\psi}_n(t) > 0,\, U_n\dot{\psi}_n(t) > U_{n'}\dot{\psi}_{n'}(t),\, \forall\, n \in S_1,\, n' \notin S_1$ [see (10)]. Therefore, $U_n\psi_n(t) > U_{n'}\psi_{n'}(t),\, \forall\, n \in S_1,\, n' \notin S_1,\, t = t_1$, where t_1, as previously defined, is the starting time of τ_2. Therefore, $n' \notin S_2$. Since $S_1 \neq S_2$, we must have $S_1 = S_2 + m$. ■

The aforementioned lemmas show that there must be a Z-SP regime with $Z = \Omega$ (Lemma 4) followed immediately by an S-SP with $S = \Omega$ (Lemma 5). The possible regimes afterwards are S-SP regimes defined in Lemma 6. We now show that an FP regime must be the last regime before the maintenance period.

Lemma 7: Let τ_1 and τ_2 be two consecutive time intervals, τ_2 following τ_1. Further, S-SP regime with its active set S is in τ_1. Then, FP regime is in τ_2 if and only if $R(S) = 2$. (Recall that $R(S)$ denotes the number of units in S).

Proof: If $R(S) > 2$ there would exist $n_1 \in S$ and $n_2 \in S$ such that $n_1 > m$ and $n_2 > m$, where $m = \min_{n \in S} n$. If FP is in τ_2 then either $u_{n_1}(t) = 0$ or $u_{n_2}(t) = 0,\, t \in \tau_2$. However, this contradicts the arguments established in the first part of Lemma 6.

If $R(S) = 2$, there exists an $n \in S,\, n > m$. According to the argument in Lemma 6, the only possible regime in $t \in \tau_2$ is an FP regime. ■

It is easy to show that only the maintenance period can stop an FP regime. The previous lemmas established the optimal sequence of regimes between the Z-SP with $Z = \Omega$ and the maintenance period. It is summarized in the following lemma.

Lemma 8: The optimal production regimes from the Z-SP regime to the maintenance period are the following: Z-SP $\rightarrow$ S-SP$_1$ $\rightarrow$ S-SP$_2$ $\rightarrow \cdots$ S-SP$_{N-1}$ $\rightarrow$ FP $\rightarrow$ Maintenance, where S-SP$_k$ is an S-SP regime with its active set being $S_k = \{k,\, k+1,\, \ldots,\, N\}$.

A similar lemma will show that the optimal production regimes after the maintenance period is the reverse of the sequence in Lemma 8 due to the agreeable cost coefficients (see Assumption 1). Maintenance $\rightarrow$ FP $\rightarrow$ S-SP$_{N-1} \cdots \rightarrow$ S-SP$_2$ $\rightarrow$ S-SP$_1$ $\rightarrow$ Z-SP.

Having determined the optimal control regime sequence, our next step is to determine t_n, the time instances at which the regimes change after Z-SP regime, but before the maintenance, and t'_n, that after the maintenance as shown in Fig. 1.

We further denote maintenance interval $[t_1^M,\, t_2^M]$, and time instance t_n^* at which inventory levels cross zero line, $n = 1, 2, \ldots, N$. By integrating (2), we immediately find

$$\left(1 - \sum_{i=n+1}^{N-1} \frac{d_i}{U_i}\right) U_n(t_{n+1} - t_n) - d_n(t_n^* - t_n) = 0$$

$$n = 1, \ldots, N,\; t_{N+1} = t_1^M \tag{21}$$

$$\left(1 - \sum_{i=n+1}^{N-1} \frac{d_i}{U_i}\right) U_n(t'_n - t'_{n+1}) - d_n(t'_n - t_n^*) = 0$$

$$n = 1, \ldots, N,\; t'_{N+1} = t_2^M. \tag{22}$$

Integrating (10), we will obtain another set of N equations

$$\sum_{i=1}^{n-1} c_i^+ U_i(t_{i+1} - t_i) + c_n^+ U_n(t_n^* - t_n)$$

$$= \sum_{i=1}^{n-1} c_i^- U_i(t'_i - t'_{i+1}) + c_n^- U_n(t'_n - t_n^*),$$

$$n = 1, \ldots, N. \tag{23}$$

$$t_{N+1} = t_1^M,\; t'_{N+1} = t_2^M.$$

The aforementioned $3N$ equations can then be used to determine the $3N$ unknown t_n, t'_n, and t_n^*.

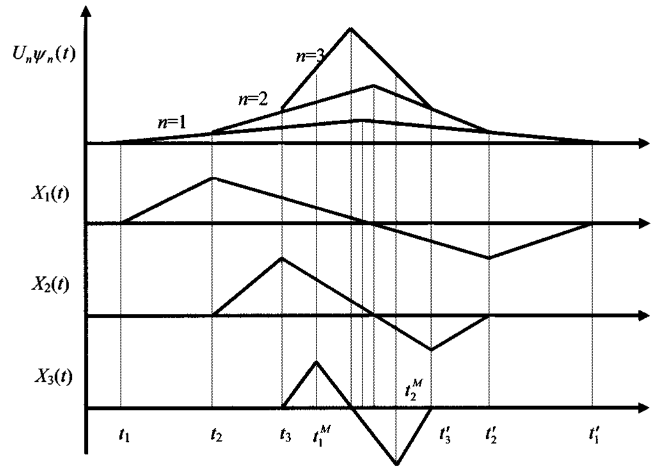

Fig. 1. Optimal behavior of the state and co-state variables for $N = 3$.

V. Scheduling Algorithm

Step 1) Sort products according to $c_n^+ U_n$ in ascending order.
Step 2) Find $3N$ switching points $t_n,\, t'_n,\, t_n^*,\, n = 1, \ldots, N$ by solving $3N$ equations (21)–(23).
Step 3) Determine the optimal production rates in each regime according to Lemmas 2 and 3.

Note that the aformentioned algorithm the production is organized according to the weighted lowest production rate (WLPR) rule, where the maximum production rate is weighted by the inventory or backlog costs. In contrast to most WLPR rules, which only allow one product with the lowest production rate to be produced at a time, this algorithm may assign a number of products to be produced concurrently. Since the production rate is inversely proportional to the production time, the concurrent WLPR rule is consistent with the weighted longest processing time (WLPT) rule well known in scheduling literature [13].

The complexity of the algorithm is determined by Step 2), which requires $O(N^3)$ time to solve.

VI. Conclusion

The optimal production control of a workstation with N-product-type and periodic maintenance is studied. The model portrays the behavior of many systems, such as telecommunications and computer complexes, which have multiprocessor ability to share resources. The objective is to minimize inventory and backlog costs. With the aid of the maximum principle, properties of the optimal regimes and conditions for solving the problem polynomially are derived. As a result, an efficient algorithm is constructed for solving the problem in $O(N^3)$ time if inventory and backlog costs are agreeable.

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Controller Synthesis for Sign-Invariant Impulse Response

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***Abstract*—In this note, we consider the problem of designing controllers for discrete-time linear time-invariant (LTI) plants that render the closed-loop impulse response nonnegative. Such systems have a nonundershooting and nonovershooting step response. We first show that the impulse response of any discrete-time LTI system changes sign at least "r" times if it has "r" real, positive zeros outside a circular disk centered at the origin and containing all its poles. We then show that a necessary and sufficient condition on the plant for the existence of a compensator that makes the closed loop impulse response sign invariant is that there be no real, positive, nonminimum phase plant zeros. Finally, we show, by construction, how such a compensator may be synthesized when the plant does satisfy the existence condition.**

***Index Terms*—Impulse response, nonminimum phase systems, nonnegative impulse response, parity interlacing, step response.**

I. INTRODUCTION

In this note, we consider the problem of designing controllers for discrete-time linear time-invariant (LTI) systems that achieve a nonnegative closed loop impulse response. We will refer to such systems having sign invariant impulse response as sign invariant systems; the corresponding transfer functions will be referred to as sign invariant transfer functions. Sign invariant systems have two features of practical interest.

1) The step response will neither undershoot nor overshoot. This feature is useful, for example, in designing control systems to fill a tank without spilling [10], or to design pick and place robots from one edge of the room to another [3].
2) The dc gain will equal its induced l_∞ gain and hence, equal any of its induced l_p gains. This feature is useful in applications such as automatic vehicle following [11], where the induced l_∞ gain must be minimized subject to an equality constraint on the dc gain.

A concise characterization of sign invariant systems does not exist in the literature. Attempts to characterize the oscillatory nature of the impulse and step response from the location of the poles and zeros of its transfer function is documented in [2], [6], [7], [15], and [17]. In [2] and [7], a class of sign invariant transfer functions is constructed by building elementary sign invariant transfer functions of first, second, and third order, and their products. In [17], there are two results of significant interest: the first result states that the impulse response of a continuous-time LTI system with r real zeros to the right of all its poles will change sign at least r times. The second result states that if the impulse response changes sign r times, then sufficiently higher order derivatives of the transfer function will have exactly r real, positive roots; furthermore, the relationship between those roots and the time at which the impulse response changes sign is explicitly characterized. While the first result in [17] indicates that any stable system with real, nonminimum phase zeros has an undershooting step response, the results in [8] and [15] further classify the undershoot in a step response into Type A or Type B based on the time of its first occurrence. In [6], an upper bound on the number of sign changes of the impulse and step response for continuous-time systems, based on the location of poles and zeros of the transfer function, is provided.

In [10], two sufficient conditions are provided for obtaining a nonovershooting step response of a continuous-time LTI system, based on state space data. While a discrete-time LTI system is sign invariant iff its Markov parameters are nonnegative, this is not necessarily true in the case of continuous-time systems. In [1], it is conjectured that the continuous-time LTI system described by the minimal triplet, $(A,\, B,\, C)$, is sign invariant iff the Markov parameters of some modified triplet, $(\lambda I + A,\, B,\, C)$ are nonnegative for some real λ. A counterexample to this conjecture is any minimal realization with $1+\epsilon-\cos(w_0 t)$ as its impulse response, where ϵ is an arbitrarily small positive number [13].

The synthesis of a nonovershooting step response via a two-parameter compensator is considered in [7] for continuous-time systems. In [3], it is shown that a nonovershooting compensator exists for any discrete-time time system. However, a nonundershooting compensator may or may not exist.

In this note, we consider the following discrete-time control system, as shown in Fig. 1. The problem considered here is to determine $G_{c1},\, G_{c2},\, G_f$ so that the closed loop is internally stable and the impulse response from the command $r(k)$ to the output $y(k)$ is nonnegative.

The organization of this note is as follows. In Section II, we present two necessary conditions for a discrete-time system to be sign invariant. These conditions are: 1) there should be no real, positive nonminimum phase zeros outside a disk centered at the origin and containing all its poles and 2) a positive, real pole of the system must have maximum

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