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Scheduling parallel machines by the dynamic newsboy problem

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Abstract

The newsboy problem is a well-known operations research model. Its various extensions have been applied to scheduling and evaluating advanced orders in manufacturing, retail and service industries.

This paper focuses on a dynamic, continuous-time generalization of the single period newsboy problem. Similar to the classical newsboy problem, the model may represent the inventory of an item that becomes obsolete quickly, spoils quickly, or has a future that is uncertain beyond a single period. The problem is characterized by a number of newsboys (machines) whose operations are organized and controlled in parallel. The objective is to minimize shortage and surplus costs occurring at the end of the period as in the classical newsboy problem, as well as intermediate production and surplus costs that are incurred at each time point along the period. We prove that this continuous-time problem can be reduced to a number of discrete-time problems which are determined by loose, balanced and pressing production conditions. As a result, a polynomial-time combinatorial algorithm is derived in order to find globally optimal solutions.

Scope and purpose

The classical, single-period newsboy problem is to find a product order quantity that either maximizes the expected profit or minimizes the expected costs of overestimating and underestimating probabilistic demand. The basic point of the classical newsboy problem is that while a decision has to be made at the beginning of a period of time there is no way to either get or use information (or updates) on the demand realization before it is accumulated, i.e., before the end of the period. This very point is adopted in the paper. The importance and applicability of such a model are widely discussed in literature. Specifically, the model may represent the inventory of an item that becomes obsolete quickly, spoils quickly, or has a future that is uncertain beyond a single period. Furthermore, the decision on inventory in the classical newsboy problem is determined in terms of the total amount to be acquired or produced over the entire planning period. In contrast, we suggest

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a dynamic continuous-time extension that enables us to make a decision at each point of time and take into account all associated costs during the planning horizon.

1. Introduction

The classical, single-period newsboy problem is to find a product order quantity that either maximizes the expected profit or minimizes the expected costs of overestimating and underestimating probabilistic demand. The newsboy problem has attracted considerable attention since the pioneering papers of Arrow et al. [1], and Morse and Kimball [2]. An extensive literature review on various extensions of the classical newsboy problem and related inventory control models can be found, for example, in Khouja [3] and Silver et al. [4]. Among the numerous extensions to this problem suggested so far, one can find different models with respect to objectives (see, for example, Chung [5] and Eeckhoudt et al. [6]); supplier pricing policies [7,8]; news-vendor pricing policies and discount structures [9,10]; random yield of defective units [11] or of production capacity [12]; multi-products [13,14] and a number of subperiods to prepare for the selling season [15–17]. The idea behind the last type of extension is that there may be many periods to produce the items, which will be sold in a single season. Such dynamic models stress the importance of timing in producing or purchasing the items. These models commonly utilize special product (or product family) and demand parameters to optimize operations under limited production capacity over each subperiod [15,17]. The former work, which deals with several families of style goods, resulted in a stochastic, mixed-integer programming problem, the latter formulated the single and multi-product cases as dynamic programming problems. Both studies suggest heuristic methods to provide an approximate solution. Matsuo [16] observed that a limitation of these works is that they include discrete production subperiods to assign production and suggested a continuous-time heuristic approach for improving the objective function value when approximating the optimal solution to the problem.

In this paper, we consider a continuous-time extension to the classical, single-period newsboy problem. Parallel newsboys process the flow of products which are then delivered to customers. As in the classical single-period newsboy problem, the demand is assumed to be unknown during the planning horizon, but the cumulative demand at the end of the planning horizon is known. The objective is to adjust the production rates during the planning horizon in order to minimize total costs. In this paper, the total costs include shortage or surplus costs occurring at the end of the planning horizon (as considered in the classical newsboy problem), as well as the surplus and production costs along the planning horizon. Note that in contrast to the classical model, the dynamic continuous-time approach enables us to make a decision at each point of time.

To study this continuous-time problem, we use the maximum principle [18], which is closely related to the widely used dynamic programming. The dynamic programming was originally developed for discrete-time problems to optimize numerically with the Hamiltonian–Jacobi–Bellman function by looking at a finite step at the future. As exact timing is required, i.e., the problem is continuous time, the step size goes to zero and the numerical search results in exponential explosion. The maximum principle is a dynamic programming formulation developed for continuous-time problems in a form of necessary optimality conditions. The advantage of such a formulation is two-fold:

(i) exact timing for decision variables can be sought for and (ii) analytical forms of the optimal solutions (rather than only numerical) may be derived.

In this paper, the dynamic, continuous-time, newsboy problem, its deterministic equivalent and a dual formulation are presented in Section 2. In Section 3 analytical properties of the optimal solutions are derived with the aid of the maximum principle. As a result, the continuous-time, parallel-newsboy problem is reduced to a number of discrete problems of sorting or ranking newsboys, determining optimal production conditions and a finite number of switching points. The advantage of this approach is that it enables us to develop a polynomial-time, combinatorial algorithm, which provides a globally optimal solution as shown in Section 4. An example and computational results are presented in Section 5. Section 6 summarizes the results.

2. Problem formulation

Since the described continuous-time newsboy problem can be straightforwardly applied to production scheduling of parallel machines, we further introduce the problem in the context of a flow-shop.

Consider a manufacturing system containing N parallel machines and a buffer located after the machines to collect finished products. The system produces a single product-type to satisfy a cumulative demand, D, for the product-type by the end of a planning horizon, T. This system can be described by the following differential equation:

$$\dot{X}(t) = \sum_n U_n u_n(t), \quad X(0) = X^0, \tag{1}$$

where $X(t)$ is the surplus level in the buffer by time t; U_n is the maximum production rate of machine n; $u_n(t)$ is the production rate of machine n at time t, X^0 is a constant. In this paper, $u_n(t)$ is the decision variable, whose binary value can be instantly set within (0,1) bounds:

$$u_n(t) = \begin{cases} 1 & \text{if machine } n \text{ produces at time } t, \\ 0 & \text{otherwise,} \end{cases} \qquad n = 1, 2, \ldots, N. \tag{2}$$

The product demand D is a random variable representing the yield amount of the product-type and characterized by probability density $\varphi(D)$ and cumulative distribution $\Phi(a) = \int_0^a \varphi(D)\,\mathrm{d}D$ functions, respectively. For each planning horizon T, there will be a single realization of D, which is known only by time T. Therefore, the decision has to be made under these uncertain conditions before the production starts.

Eq. (1) presents the flow of products through the machines and buffer. The flow is determined by the total production rate of all machines engaged in production. The difference between the cumulative production and the cumulative demand, $X(T) - D$, is the surplus level. If the cumulative demand exceeds the cumulative production, i.e., if the surplus is negative, a penalty will have to be paid for the lost sales. On the other hand, if $X(T) - D > 0$, overproduction cost is incurred at the end of the planning horizon. Furthermore, production costs are incurred at points t when machines are not idle and inventory holding costs are incurred at points when buffer levels are positive, $X(t) > 0$. Note, that (1) implies that $X(t) \geqslant 0$ always holds.

The objective is to find such production rates $u_n(t)$ that satisfy constraints (1)–(2) while minimizing the following expected cost over the planning horizon T:

$$J = E\left[\int_0^T \left(\sum_n c_n u_n(t) + hX(t)\right) \mathrm{d}t + P(X(T) - D)\right] \to \min, \tag{3}$$

where c_n is the production cost of machine n per time unit, h is the inventory holding cost of one product per time unit, and piece-wise linear cost functions are used for the surplus/backlog costs,

$$P(Z) = p^+Z^+ + p^-Z^-, \tag{4}$$

where $Z^+ = \max\{0, Z\}$, $Z^- = \max\{0, -Z\}$, p^+ and p^- are the costs of one product surplus and shortage, respectively.

Let us substitute (4) into objective (3). Then, given probability density $\varphi(D)$ of the demand, we find

$$\begin{aligned} J &= \int_0^T \left(\sum_n c_n u_n(t) + hX(t)\right) \mathrm{d}t \\ &\quad + \int_0^\infty p^+ \max\{0, X(T) - D\}\varphi(D)\,\mathrm{d}D + \int_0^\infty p^- \max\{0, D - X(T)\}\varphi(D)\,\mathrm{d}D \\ &= \int_0^T \left(\sum_n c_n u_n(t) + hX(t)\right) \mathrm{d}t + \int_0^{X(T)} p^+(X(T) - D)\varphi(D)\,\mathrm{d}D \\ &\quad + \int_{X(T)}^\infty p^-(D - X(T))\varphi(D)\,\mathrm{d}D. \end{aligned} \tag{5}$$

The new objective (5) is subject to constraints (1)–(2), which together constitute a deterministic problem equivalent to the stochastic problem (1)–(4).

We use the maximum principle to study the equivalent deterministic problem and formulate a dual problem [18]. The Hamiltonian is the objective for the dual problem, which according to the maximum principle is maximized for each t by the optimal decision variables $u_n(t)$. Similar to the dynamic programming, the Hamiltonian is constructed from the time-dependent part of the primal objective function (5) and the right-hand side of the dynamic equation (1)

$$H(t) = -\sum_n c_n u_n(t) - hX(t) + \psi(t)\sum_n U_n u_n(t) \to \max,$$

where the multiplier $\psi(t)$ is referred to as a costate variable. The costate variable measures the dynamic marginal cost which is the change in the objective function value resulting from a unit change of $X(t)$ at time t. According to the maximum principle, $\psi(t)$ satisfies the following dual (costate) equation $\dot{\psi}(t) = -\partial H(t)/\partial X(t)$, i.e.,

$$\dot{\psi}(t) = h \tag{6}$$

with transversality (boundary) constraint

$$\psi(T) = -\frac{\partial[\int_0^{X(T)} p^+(X(T)-D)\varphi(D)\,\mathrm{d}D + \int_{X(T)}^{\infty} p^-(D-X(T))\varphi(D)\,\mathrm{d}D]}{\partial X(T)}$$

$$= -\int_0^{X(T)} p^+\varphi(D)\,\mathrm{d}D + \int_{X(T)}^{\infty} p^-\varphi(D)\,\mathrm{d}D,$$

i.e.,

$$\psi(T) = -p^+\Phi(X(T)) + p^-(1-\Phi(X(T))). \tag{7}$$

By rearranging only decision-variable-dependent terms of the Hamiltonian we obtain

$$H_u(t) = \sum_n (U_n\psi(t) - c_n)u_n(t). \tag{8}$$

Since this term is linear in $u_n(t)$, it can be easily verified that the optimal production rate that maximizes the Hamiltonian is

$$u_n(t) = \begin{cases} 1 & \text{if } \psi(t) > \frac{c_n}{U_n}, \\ w \in \{0,1\} & \text{if } \psi(t) = \frac{c_n}{U_n}, \\ 0 & \text{if } \psi(t) < \frac{c_n}{U_n}. \end{cases} \tag{9}$$

Thus under the optimal solution, the nth machine can be idle ($\psi(t) < c_n/U_n$), working at its maximum production rate ($\psi(t) > c_n/U_n$), or entering the singular regime ($\psi(t) = c_n/U_n$) which is characterized by infinite switching between 0 and 1.

3. Properties of the optimal solution

We next study the basic properties of the optimal solution. The first property is the so-called integrality property, which is due to the fact that the singular regime never exists on an optimal trajectory as proven in the following lemma.

Lemma 1. *Given that constraint (2) is relaxed as* $0 \leqslant u_n(t) \leqslant 1$, $n = 1,2,\ldots,N$ *and* $h \neq 0$, *there always exists an optimal solution, such that* $u_n(t)$ *is equal to either* 1 *or* 0 *at each measurable interval of time.*

Proof. The proof is by contradiction. According to the optimality condition (9), the singular regime is the only regime along which infinite switching of the decision variable between 0 and 1 is possible at a measurable time interval, τ. Assuming the singular regime condition $\psi(t) = c_n/U_n$ holds over τ and differentiating this condition, we find

$$\dot{\psi}(t) = 0,$$

which contradicts the costate equation (6), $\dot{\psi}(t) = h \neq 0$. □

Based on the integrality property, the unimodality of the optimal solutions is shown in the next lemma.

Lemma 2. *Let* $\partial\Phi(X(T))/\partial X(T) \neq 0$. *Problems* (1)–(2), (5) *is unimodal, i.e., there is only one optimal value for the objective function.*

Proof. First of all, note that binary constraints (2) can be relaxed as stated in Lemma 1. Eq. (1) is linear. The objective function (5) consists of three terms. The first two terms are linear as well. The third term

$$R = \int_0^{X(T)} p^+(X(T) - D)\varphi(D)\,\mathrm{d}D + \int_{X(T)}^{\infty} p^-(D - X(T))\varphi(D)\,\mathrm{d}D$$

is strictly convex with respect to $X(T)$, because $\partial^2 R/\partial X(T)^2 = (p^+ + p^-)\partial\Phi(X(T))/\partial X(T) > 0$. Thus, problem are (1)–(2), (5) unimodal. □

Since the primal problem is unimodal, the maximum principle provides not only the necessary, but also the sufficient conditions of optimality. Therefore, all triplets $(u_n(t), X(t), \psi(t))$ that satisfy primal (1)–(2), dual (6)–(7), and (9) will minimize the objective function (3).

The next lemma shows optimal sequencing of the machines, i.e., the order in which it is optimal for the machines to switch on for production.

Lemma 3. *Given that machine* $n2$ *is switched on after machine* $n1$, *the following holds*:

$$\frac{c_{n1}}{U_{n1}} \leqslant \frac{c_{n2}}{U_{n2}}.$$

Proof. The proof immediately follows from optimality condition (9), Lemma 1 and the fact that the costate variable is continuous, increasing in time function as defined by Eqs. (6). □

Note that using the same argument as in the proof of Lemma 3, we can conclude that there is no preemption in the system. That is, if a machine, n, is switched on at a time, t_n, then the optimality condition $\psi(t) > c_n/U_n$ will hold for $t_n < t \leqslant T$ and, thus, the machine will not be switched off before the end of the planning horizon.

Henceforth, without loss of generality, we assume that all machines are ordered and numbered in increasing order of c_n/U_n.

Given optimal sequencing, we now use a constructive approach to solve the problem. That is, we first propose a solution, which satisfies the optimality conditions (9), and then we show that this solution is feasible and, therefore, indeed optimal. The following three lemmas study three different types of optimal behavior. Specifically, Lemma 4 is devoted to the case characterized by balanced production conditions. This implies that the relationship between the initial inventories, demand, maximum production rate, cost and planning horizon length is such that the system has enough time to initiate the production from a point of time, which is not necessarily at the beginning of the planning horizon. Lemma 5 presents the case of loose production conditions when it is optimal not to produce at all. Finally, Lemma 6 studies pressing production conditions when it is optimal to start the production from the very beginning of the planning horizon.

Lemma 4. *Given K, $1 \leqslant K \leqslant N$, define time t_1 to satisfy*

$$-p^{+}\Phi\left(X^0+\sum_{n=1}^{K}U_n\left(T-\frac{c_n}{hU_n}+\frac{c_1}{hU_1}-t_1\right)\right)$$
$$+p^{-}\left(1-\Phi\left(X^0+\sum_{n=1}^{K}U_n\left(T-\frac{c_n}{hU_n}+\frac{c_1}{hU_1}-t_1\right)\right)\right)$$
$$=\frac{c_1}{U_1}+h(T-t_1). \tag{10}$$

If $0 \leqslant t_1$, $t_K < T$, then $t_n = 1/h(c_n/U_n - c_1/U_1) + t_1$ for $n = 2,\dots,K$ and the optimal solution is given by: $u_n(t) = 0$ *for* $0 \leqslant t < t_n$; $u_n(t) = 1$ *for* $t_n \leqslant t \leqslant T$, $n = 1,\dots,K$.

Proof. For primal (1) and dual (6) equations, consider the following solution which is determined by K switching points and satisfies the optimality conditions (9):

$$u_n(t)=0 \quad \text{for} \quad 0 \leqslant t < t_n, \qquad u_n(t)=1 \quad \text{for} \quad t_n \leqslant t \leqslant T, \quad n=1,\dots,K,$$

$$X(T)=X^0+\sum_{n=1}^{K}U_n(T-t_n), \tag{11}$$

$$\psi(t)=\psi(t_1)+h(t-t_1), \quad t \geqslant t_1, \qquad \psi(T)=\frac{c_1}{U_1}+h(T-t_1),$$

$$\psi(t_n)=\frac{c_n}{U_n}, \quad n=1,\dots,K. \tag{12}$$

If this solution is feasible, then according to Lemma 2 it is optimal. To verify the feasibility, we first determine the switching points $t_n = 1/h(c_n/U_n - c_1/U_1) + t_1$, $n > 1$ from (12) and by substituting them into (11) obtain

$$X(T)=X^0+\sum_{n=1}^{K}U_n\left(T-\frac{c_n}{hU_n}+\frac{c_1}{hU_1}-t_1\right). \tag{13}$$

Next, by taking into account the transversality condition (7), we find

$$-p^{+}\Phi(X(T))+p^{-}(1-\Phi(X(T)))=\frac{c_1}{U_1}+h(T-t_1). \tag{14}$$

Finally, by substituting $X(T)$ from (13) into (14), we determine Eq. (10) in unknown t_1. The feasibility of this solution is ensured by $0 \leqslant t_1 < T$ as stated in the lemma. □

To illustrate the balanced production conditions derived in Lemma 4, consider a production system characterized by $K = 2$ and the uniform demand distribution

$$\varphi(D)=\begin{cases}\frac{1}{d} & \text{for } 0 \leqslant D \leqslant d\\ 0 & \text{otherwise}\end{cases} \qquad \text{and} \qquad \Phi(a)=\frac{a}{d}.$$

Then from (10) we have

$$-\frac{p^+}{d}\left(X^0+U_2\left(T-\frac{c_2}{hU_2}+\frac{c_1}{hU_1}-t_1\right)+U_1(T-t_1)\right)$$
$$+p^- -\frac{p^-}{d}\left(X^0+U_2\left(T-\frac{c_2}{hU_2}+\frac{c_1}{hU_1}-t_1\right)+U_1(T-t_1)\right)=\frac{c_1}{U_1}+h(T-t_1),$$

i.e.,

$$T-\frac{p^- -(\frac{p^+}{d}+\frac{p^-}{d})X^0-(\frac{p^+}{d}+\frac{p^-}{d})U_2(\frac{c_1}{hU_1}-\frac{c_2}{hU_2})-\frac{c_1}{U_1}}{h+(\frac{p^+}{d}+\frac{p^-}{d})(U_1+U_2)}=t_1 \quad \text{and}$$

$$t_2=\frac{1}{h}\left(\frac{c_2}{U_2}-\frac{c_1}{U_1}\right)+t_1.$$

Thus,

$$\text{if } T-\frac{p^- -(\frac{p^+}{d}+\frac{p^-}{d})X^0-(\frac{p^+}{d}+\frac{p^-}{d})U_2(\frac{c_1}{hU_1}-\frac{c_2}{hU_2})-\frac{c_1}{U_1}}{h+(\frac{p^+}{d}+\frac{p^-}{d})(U_1+U_2)}\geqslant 0 \quad \text{and}$$

$$t_2=\frac{1}{h}\left(\frac{c_2}{U_2}-\frac{c_1}{U_1}\right)+t_1<T,$$

then $u_1(t)=0$ for $0\leqslant t<t_1$; $u_1(t)=1$ for $t_1\leqslant t\leqslant T$, and $u_2(t)=0$ for $0\leqslant t<t_2$; $u_2(t)=1$ for $t_2\leqslant t\leqslant T$.

Note, that if $K=1$, then these conditions simplify to

$$0\leqslant T-\frac{p^- -(p^+/d+p^-/d)X^0-c_1/U_1}{h+(p^+/d+p^-/d)U_1}<T.$$

This implies that in order for the balanced production conditions to hold for at least one machine, it is necessary that $p^- -(p^+/d+p^-/d)X^0-c_1/U_1>0$. Furthermore, if $X^0=0$, then this condition is, $p^->c_1/U_1$, i.e., the balanced production can be profitable only if one product backlog is more expansive than the production cost related to the production rate of the machine.

Lemma 5. *Define time t_1 to satisfy*

$$-p^+\Phi(X^0+U_1(T-t_1))+p^-(1-\Phi(X^0+U_1(T-t_1))=\frac{c_1}{U_1}+h(T-t_1). \tag{15}$$

If $t_1\geqslant T$, then it is optimal not to produce any product, i.e., $u_n(t)=0$ for $0\leqslant t\leqslant T$.

Proof. The proof is very similar to that of Lemma 4 and is due to substituting $K=1$ into Eq. (10) which results in Eq. (15). Then the condition $t_1\geqslant T$ implies that it is not optimal for even one machine ($K=1$) to switch on along the planning horizon. □

To illustrate the loose production conditions, we proceed with the example used for illustrating the balanced conditions. Then from (15) we have

$$-\frac{p^+}{d}(X^0 + U_1(T - t_1) + p^- - \frac{p^-}{d}(X_0 + U_1(T - t_1)) = \frac{c_1}{U_1} + h(T - t_1),$$

i.e.,

$$\frac{p^- - (\frac{p^+}{d} + \frac{p^-}{d})X^0 - \frac{c_1}{U_1}}{h + (\frac{p^+}{d} + \frac{p^-}{d})U_1} = T - t_1.$$

Thus, it is not optimal to produce at all if $p^- - (p^+/d + p^-/d)X^0 - c_1/U_1 \leqslant 0$. It is easy to observe, that this condition complements the corresponding condition obtained for the balanced production conditions. Similarly, if $X^0 = 0$, then the no production condition takes a simple form, $p^- \leqslant c_1/U_1$, i.e., the backlog cost is less than or equal to the production cost related to the production rate of the first machine. Recall that the machines have to be ordered so that $c_1/U_1 = \min_n\{c_n/U_n\}$.

Remark 1. Eq. (15) allows us to determine optimal production or order quantity $X(T)$ when at most one machine ($K = 1$) is producing. Note that by setting inventory h and production c_1 costs at zero in Eq. (15), one can now obtain (well known in the operations research literature) economic order quantity with limited sales period for the classical, single-period newsboy problem: $\Phi(X(T)) = p^-/(p^+ + p^-)$.

Lemma 6. *Given L and K, $1 \leqslant L \leqslant K \leqslant N$, define costate $\psi(0)$ to satisfy*

$$-p^+\Phi\left(X^0 + \sum_{n=L}^{L} U_n T + \sum_{n=L+1}^{K} U_n\left(T - \frac{c_n}{hU_n} + \psi(0)\right)\right)$$
$$+ p^-\left(1 - \Phi\left(X^0 + \sum_{n=L}^{L} U_n T + \sum_{n=L+1}^{K} U_n\left(T - \frac{c_n}{hU_n} + \psi(0)\right)\right)\right) = \psi(0) + hT \tag{16}$$

and time points t_n, $n = L+1,\ldots,K$ to satisfy $t_n = 1/h(c_n/U_n - \psi(0))$.

If $c_L/U_L \leqslant \psi(0) < c_{L+1}/U_{L+1}$ and $t_K < T$, then the optimal solution is given by

$$u_n(t) = 0 \quad \text{for } 0 \leqslant t < t_n, \qquad u_n(t) = 1 \quad \text{for } t_n \leqslant t \leqslant T, \qquad n = L+1,\ldots,K,$$

$$u_n(t) = 1 \quad \text{for } 0 \leqslant t \leqslant T, \quad n = 1,\ldots,L.$$

Proof. For primal (1) and dual (6) equations, consider the following solution which is determined by K–L switching points and satisfies the optimality conditions (9)

$$u_n(t) = 0 \quad \text{for } 0 \leqslant t < t_n, \qquad u_n(t) = 1 \quad \text{for } t_n \leqslant t \leqslant T, \qquad n = L+1,\ldots,K,$$
$$u_n(t) = 1 \quad \text{for } 0 \leqslant t \leqslant T, \qquad n = 1,\ldots,L,$$

$$X(T) = X^0 + \sum_{n=1}^{L} U_n T + \sum_{n=L+1}^{K} U_n(T - t_n), \tag{17}$$

$$\psi(t) = \psi(0) + ht, \qquad \psi(t_n) = \frac{c_n}{U_n}, \quad n = L+1, \dots, K. \tag{18}$$

If this solution is feasible, then, according to Lemma 2, it is optimal. To verify the feasibility, we first determine the switching points $t_n = 1/h(c_n/U_n - \psi(0))$, $n = L+1, \dots, K$ from (18) and by substituting them into (17) obtain

$$X(T) = X^0 + \sum_{n=1}^{L} U_n T + \sum_{n=L+1}^{K} U_n \left(T - \frac{c_n}{hU_n} + \psi(0) \right). \tag{19}$$

Next, by taking into account the transversality condition (7), we find

$$-\, p^+ \Phi(X(T)) + p^-(1 - \Phi(X(T))) = \psi(0) + hT. \tag{20}$$

Finally, by substituting $X(T)$ from (19) into (20), we determine Eq. (16) in unknown $\psi(0)$. The feasibility of this solution is ensured by

$$\frac{c_L}{U_L} \leqslant \psi(0) < \frac{c_{L+1}}{U_{L+1}} \tag{21}$$

and $t_n < T$ as stated in the lemma. □

To illustrate the pressing production conditions with the same example, we consider the case of $L = 1$ (the first machine works from the very beginning of the planning horizon, $t_1 = 0$) and $K = 2$ (the second machine works from $t_2 < T$). Then from (16) we find

$$-\frac{p^+}{d}\left(X^0 + U_1 T + U_2 \left(T - \frac{c_2}{hU_2} + \psi(0) \right) \right)$$
$$+ p^- - \frac{p^-}{d}\left(X^0 + U_1 T + U_2 \left(T - \frac{c_2}{hU_2} + \psi(0) \right) \right) = \psi(0) + hT,$$

i.e.,

$$\psi(0) = \frac{p^- - hT - (\frac{p^+}{d} + \frac{p^-}{d})(X^0 + (U_1 + U_2)T - \frac{c_2}{h})}{1 + (\frac{p^+}{d} + \frac{p^-}{d})U_2} \quad \text{and} \quad t_2 = \frac{1}{h}\left(\frac{c_2}{U_2} - \psi(0) \right).$$

Therefore, the pressing production conditions take the following form:

If

$$\frac{c_1}{U_1} \leqslant \frac{p^- - hT - (\frac{p^+}{d} + \frac{p^-}{d})(X^0 + (U_1 + U_2)T - \frac{c_2}{h})}{1 + (\frac{p^+}{d} + \frac{p^-}{d})U_2} < \frac{c_2}{U_2} \quad \text{and} \quad \frac{1}{h}\left(\frac{c_2}{U_2} - \psi(0) \right) < T,$$

then $u_2(t) = 0$ for $0 \leqslant t < t_2$, $u_2(t) = 1$ for $t_2 \leqslant t \leqslant T$ and $u_1(t) = 1$ for $0 \leqslant t \leqslant T$.

4. Algorithm

The algorithm is straightforward. It verifies which of the three possible production conditions studied in Lemmas 4–6 is met and sets the corresponding optimal solution. When the number of machines which work from the very beginning of the planning horizon, L, and the maximum number of machines participating in production, K, are unknown, the algorithm enumerates for all possible combinations of L and K as described below.

INPUT: N; c_n, U_n, $n = 1,\ldots,N$; p^+, p^-, h.

Loose Production Conditions

Step 1. Find $n1 = \arg\min_n \frac{c_n}{U_n}$. Check conditions of Lemma 5 for machine $n1$. If they hold, then it is optimal not to produce at all, otherwise go to the next step.
Step 2. Sort and renumber the machines in non-decreasing order of c_n/U_n, $n = 1,\ldots,N$.

Balanced Production Conditions

Step 3. Set $K = N$. Check conditions of Lemma 4. If they are met, set the optimal solution as shown in Step 6, otherwise go to the next step.
Step 4. If $t_1 < 0$, then go to Step 7. If $t_1 > T$ then go to the next step.
Step 5. Set $K = K - 1$.
Step 6. Check conditions of Lemma 4. If they do not hold, then go to Step 5.
Otherwise calculate the switching points $t_n = 1/h(c_n/U_n - c_1/U_1) + t_1$ for $n = 2,\ldots,K$ and for $n = 1$ by (10). Set the optimal solution as: $u_n(t) = 0$ for $0 \leqslant t < t_n$; $u_n(t) = 1$ for $t_n \leqslant t \leqslant T$, $n = 1,\ldots,K$.

Pressing Production Conditions

Step 7. Set $L = 1$, $K = 0$.
Step 8. If $K > N - 1$, then set $L = L + 1$ and $K = L$ and go to the next step. Otherwise set $K = K + 1$.
Step 9. Check conditions of Lemma 6. If they do not hold, go to Step 8. Otherwise calculate $\psi(0)$ by (16) and the switching points, $t_n = 1/h(c_n/U_n - \psi(0))$, $n = L+1,\ldots,K$. Set the optimal solution as: $u_n(t) = 0$ for $0 \leqslant t < t_n$, $u_n(t) = 1$ for $t_n \leqslant t \leqslant T$, $n = L+1,\ldots,K$; $u_n(t) = 1$ for $0 \leqslant t \leqslant T$, $n = 1,\ldots,L$.
OUTPUT: Optimal $u_n(t)$ for $0 \leqslant t \leqslant T$, $n = 1,2,\ldots,N$.

Theorem 1. *Given solutions of Eqs.* (10), (15) *and* (16), *problems* (1)–(4) *is solvable in* $O(N^3)$ *time. Specifically,*

- *if given* t_1 *which satisfies Eq.* (15) *and* $t_1 \geqslant T$, *then problems* (1)–(4) *is solvable in* $O(N)$ *time*;
- *if given* t_1 *which satisfies Eq.* (10) *and* $0 \leqslant t_1 < T$, *then problems* (1)–(4) *is solvable in* $O(N^2)$ *time*;
- *if given* $\psi(0)$ *which satisfies Eq.* (16) *for* L *and* K *such that* $c_L/U_L \leqslant \psi(0) < c_{L+1}/U_{L+1}$ *and* $t_K < T$, *then problems* (1)–(4) *is solvable in* $O(N^3)$ *time.*

Proof. According to Lemmas 1–6, the algorithm finds a feasible solution which satisfies the necessary and sufficient optimality conditions. Therefore, this solution is globally optimal.

Provided a feasible solution of the optimal order quantity Eq. (15) (*loose production conditions*), Step 1 evidently results in an optimal solution of $O(N)$ complexity. Steps 3–6 treat the *balanced production conditions*. Although Step 6 requires only $O(N)$ time, it is repeated at most N times to determine the maximum number of machines, K, which participate in production. Therefore, the total estimate for this case is $O(N^2)$ time. Finally, the worst-case estimate encountered for problems (1)–(4) is due to the *pressing production conditions* (Steps 7–9). If a feasible solution of the optimal order quantity Eq. (16) can be found in $O(1)$ time, Step 9 (of $O(N)$ complexity) is repeated in double loop (at most $N^2/2$ times) to determine the number of machines which produce from the very beginning of the planning horizon, L, and the maximum number of not idle machines K. □

Remark 2. Theorem 1 estimates the complexity of solving problems (1)–(4) provided optimal order Eqs. (10), (15) and (16) can be resolved analytically. However, an analytical solution is not always available. One can readily observe that Eqs. (10), (15) and (16) are monotone in their unknowns, which implies they can be easily solved numerically to any desired precision by simple methods for finding the root of a monotone function (e.g., the dichotomous search, the golden section or the Fibonacci search).

5. Example and computational results

To illustrate each step of the algorithm, we consider a small, five-machine parallel production system. The input data for such a system is presented in Table 1.

In addition, $T=5$ time units; $X^0=0$ product units; the demand distribution is uniform, $d=24$ product units; $p^+=1\$$ per product unit; $p^-=2\$$ per product unit; and $h=0.1\$$ per product unit and time unit.

The algorithm starts by verifying whether the case of the *loose production conditions* is optimal. This is accomplished at Step 1 by determining $n1=\arg\min_n c_n/U_n=3$ and finding t_3 for machine 3 to switch on. From Eq. (15) we obtain

$$-1\left(\frac{1(5-t_3)}{24}\right)+2\left[1-\frac{1(5-t_3)}{24}\right]=0.005+0.1(5-t_3)\Rightarrow t_3=-3.87.$$

Since the obtained solution is not feasible, $t_3=-3.87<0$, the algorithm proceeds to verify whether the case of the *balanced production conditions* is optimal. Step 2 sorts and renumbers the machines in non-decreasing order of $c_n/U_n,\ n=1,\ldots,5$ as shown in Table 1. That is, the optimal ordering of the machines is: 3-5-1-2-4 (to make the presentation clearer, we will use the original machine numbers).

Next, at Steps 3–6, the algorithm finds the optimal number of working machines K by: setting $K=5,4,\ldots,1$ and solving Eq. (10) for t_3 and verifying its feasibility. For $K=2$ (i.e., machines 3 and 5 with respect to the optimal order), we obtain from Eq. (10)

$$-1\left(\frac{1(5-t_3)}{24}+\frac{6(5-t_3)-\frac{0.01}{0.1}+\frac{0.05}{0.1}}{24}\right)+2\left[1-\frac{1(5-t_3)}{24}+\frac{6(5-t_3)-\frac{0.01}{0.1}+\frac{0.05}{0.1}}{24}\right]$$
$$=0.005+0.1(5-t_3)\Rightarrow t_3=2.685.$$

Table 1
Parameters of the five-machine system

Parameters	Machine 1	Machine 2	Machine 3	Machine 4	Machine 5
U_n	2	5	1	2	6
c_n	0.04	0.15	0.005	0.08	0.06
$\frac{c_n}{U_n}$	0.02	0.03	0.005	0.04	0.01
Order	3	4	1	5	2

Since, $0 \leqslant t_3 < T = 5$ is feasible, the found solution is optimal. Therefore, at Step 6, the algorithm calculates $t_5 = 1/h(c_5/U_5 - c_3/U_3) + t_3 = 1/0.1(0.01 - 0.005) + 2.685 = 2.735$ and sets the optimal solution as

$$u_3(t) = 0 \quad \text{for } 0 \leqslant t < 2.685, \qquad u_3(t) = 1 \quad \text{for } 2.685 \leqslant t \leqslant 5,$$

$$u_5(t) = 0 \quad \text{for } 0 \leqslant t < 2.735, \qquad u_5(t) = 1 \quad \text{for } 2.735 \leqslant t \leqslant 5,$$

$$u_1(t) = 0, \quad u_2(t) = 0, \quad u_4(t) = 0 \quad \text{for } 0 \leqslant t \leqslant 5.$$

Finally, to compare the results with a more classical discrete-time approach, we choose M equally distributed time points, $t_{m+1} = t_m + \Delta$, $m = 1, 2, \ldots, M$, $t_0 = 0$, $t_M = T$ and present a straightforward, discrete-time formulation of problem (1), (2) and (5)

$$X(t_{m+1}) - X(t_m) = \Delta \sum_n U_n u_n(t_m), X(t_0) = X^0, \quad n = 1, 2, \ldots, N, \quad m = 1, 2, \ldots, M-1, \tag{22}$$

$$u_n(t_m) = \begin{cases} 1 & \text{if machine } n \text{ produces at time } t_m, \quad n = 1, 2, \ldots, N, \\ 0 & \text{otherwise}, \quad m = 0, 1, 2, \ldots, M-1, \end{cases} \tag{23}$$

$$J_M = \Delta \sum_m \left(\sum_n c_n u_n(t_m) + hX(t_m) \right) + \int_0^{X(t_M)} p^+ (X(t_M) - D)\varphi(D)\,\mathrm{d}D$$
$$+ \int_{X(t_M)}^{\infty} p^- (D - X(t_M))\varphi(D)\,\mathrm{d}D \to \min. \tag{24}$$

With respect to the uniform distribution the objective function J_M takes the following quadratic form:

$$J_M = \Delta \sum_m \left(\sum_n c_n u_n(t_m) + hX(t_m) \right) + p^+ \frac{X^2(t_M)}{2d} + p^- \frac{(d - X(t_M))^2}{2d} \to \min. \tag{25}$$

Note, that the best available approach for convex programming is interior-point based and requires $O(N^6 M^6)$ number of operations [19] to solve problems (22), (23) and (24), while the Simplex method is characterized by an exponential worst case complexity even for linear programming.

Table 2
Computational results

	GAMS, NLP-based solution					Maximum principle-based algorithm
$N = 20$						
M	25	50	100	200	500	—
Computation time (s)	0.5	2	6	35	192	1
Relative deviation (%)	43.2	17.8	11.5	6.7	4.1	0
$N = 40$						
M	25	50	100	200	500	—
Computation time (s)	1.5	7	22	85	812	2
Relative deviation (%)	79	50	21	12	5.6	0

To illustrate the theoretical estimates derived in the paper, the same problems were solved by the suggested maximum principle-based algorithm and straightforward convex programming of (22), (23) and (25) with GAMS software. Table 2 presents the results based on more than a hundred runs for $T = 200$, $N = 20$ and 40. The comparison of the computation time for the two methods and relative deviation $R = (J_M - J)/J100$ from the optimal solution J found by the suggested algorithm are obtained on PC-Pentium-III-733. The system parameters are generated randomly in the range of [0.01, 5] for inventory, backlog and surplus costs and in the range of [1, 9] for production rates.

From Table 2, one can observe that the maximum principle-based, continuous-time approach provides an optimal solution in virtually no time. On the other hand, when $N = 40$ an improvement from 12% relative deviation (rough accuracy) of the objective function to 5.6% deviation (moderate accuracy) with the discrete-time, GAMS-based approach necessitates an increase of almost 100 times in the computational burden.

6. Conclusion

A dynamic, continuous-time extension of the well-known, single-period newsboy problem is introduced to incorporate such important factors as time, multiple newsboys and controllable operation rates. The problem is presented in the context of a manufacturing system consisting of parallel machines and a buffer placed after them. With the aid of the maximum principle, the continuous-time problem is reduced to ranking machines and defining whether the production conditions are loose, balanced, or pressing. Based on this, a limited number of switching time points is calculated and production rates are assigned over the switching points with respect to the machine ranks. We show that if the equations derived for the optimal production order quantity can be solved analytically and the system operates under loose production conditions, then the original problem is solvable in $O(N)$ time. If the conditions are balanced it is solvable in $O(N^2)$ time, and if the conditions are pressing in $O(N^3)$ time. Furthermore, even if an optimal order equation is not solvable analytically, it is always solvable in polynomial time numerically to any required precision.

References

[1] Arrow KA, Harris TE, Marschak J. Optimal inventory policy. Econometrica 1951;19:250–72.
[2] Morse MP, Kimbal GE. Methods of operations research. Cambridge, MA: MIT Press, 1951.
[3] Khouja M. The single period (news-vendor) inventory problem: a literature review and suggestions for future research. Omega 1999;27:537–53.
[4] Silver EA, Pyke DF, Peterson R. Inventory management and production planning and scheduling. New York: Wiley, 1998.
[5] Chung K. Risk in inventory models: the case of the newsboy problem, optimality conditions. Journal of Operational Research Society 1990;41:173–6.
[6] Eeckhoudt L, Gollier C, Schlesinger H. The risk averse (and prudent) newsboy. Management Science 1995;41: 786–94.
[7] Kabak I, Weinberg C. The generalized newsboy problem, contract negotiations and secondary vendors. IIE Transactions 1972;4:154–7.
[8] Lin C, Kroll DE. The single item newsboy problem with dual performance measures and quantity discounts. European Journal of Operational Research 1997;100:562–5.
[9] Khouja M. The newsboy problem under progressive multiple discounts. European Journal of Operational Research 1995;84:458–66.
[10] Lau A, Lau H. The newsboy problem with price-dependent demand distribution. IIE Transactions 1988;20:168–75.
[11] Henig M, Gerchak Y. The structure of periodic review policies in the presence of random yield. Operations Research 1990;38:634–43.
[12] Ciarolo FW, Akella R, Morton TE. A periodic review, production planning model with uncertain demand-optimality of extended myopic policies. Management Science 1994;40:320–32.
[13] Lau A, Lau H. The newsstand problem: a capacitated multi-product single period inventory problem. Operations Research 1996;94:29–42.
[14] Chang PL, Lin CT. On the effect of centralization of the expected costs in a multi-location newsboy problem. Journal of Operational Research Society 1991;42:1025–30.
[15] Hausman WH, Peterson R. Multiproduct production scheduling for style goods with limited capacity, forecast, revisions, and terminal delivery. Management Science 1972;18:370–83.
[16] Matsuo H. A stochastic sequencing problem for style goods with forecast revisions and hierarchical structure. Management Science 1990;36:332–47.
[17] Bitran GR, Haas EA, Matsuo H. Production planning of style goods with high setup costs and forecast revisions. Operations Research 1986;34:226–36.
[18] Maimon O, Khmelnitsky E, Kogan K. Optimal flow control in manufacturing systems: production planning and scheduling. Boston: Kluwer Academic Publisher, 1998.
[19] Karmarkar N. A new polynomial-time algorithm for linear programming. Combinatorica 1984;4:373–95.

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